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UNIFROM NETWORK TRANSFORMARION

FOR SPATIAL ANALYSIS

ON A HETEROGENEOUS NETWORK

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Abstract

This paper proposes a general method for analyzing the distribution of points on a ‘heterogeneous’ network. First, the paper formulates a method that transforms a heterogeneous network (i.e. a network in which a probability of a point being placed in a unit line segment on the network varies according to the location of the unit line segment) into a homogeneous network (i.e. a network in which a probability of a point being placed in a unit line segment on the network is the same regardless of the location of the unit line segment). Second, the paper proves that this transformed network can be geometrically realized on a two-dimensional space (plane). These results have very useful implications. First, a heterogeneous network can be easily treated through transforming it into a homogeneous network. This contrasts to the fact that there is no method for transforming a heterogeneous plane into a homogeneous plane. Second, any methods assuming a homogeneous network can be applied to the analysis on a heterogeneous network through the uniform network transformation. Hence, it is not necessary to develop special methods for a heterogeneous network.

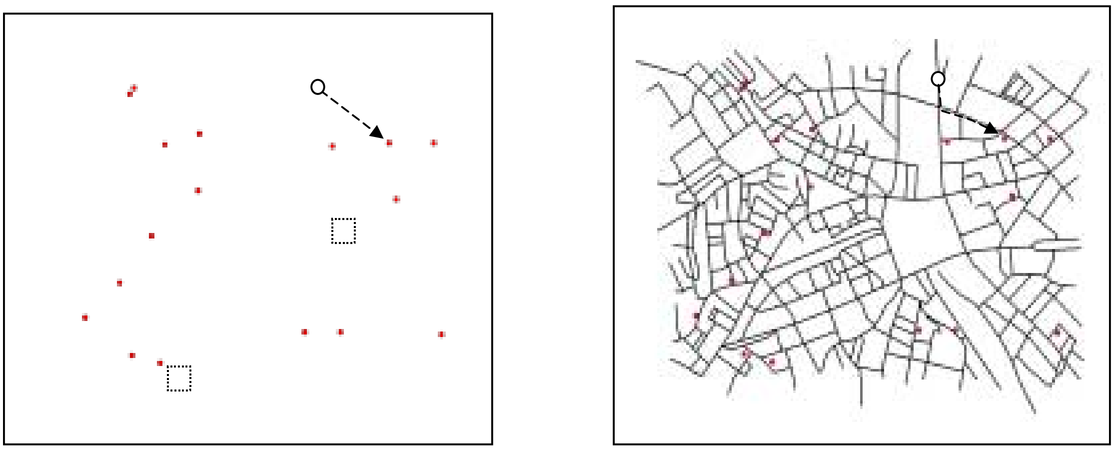
1. Introduction

We notice in the related literature (for example, King, 1969; Getis and Boots, 1976; Lewis, 1977; Bailey and Gatrell, 1995) that almost all quantitative methods for spatial analysis assume that:

i) space, S, is represented by two-dimensional space, i.e. a plane (Figure 1a),

ii) distance in S is measured with Euclidean distance (the arrowed line in Figure 1a),

iii) space S is homogeneous, i.e. the characteristics of a unit area in S are the same regardless of the location of the unit area in S (the squares indicated by the broken lines in Figure 1a); in terms of stochastic point processes, the probability of a point being placed in a unit area in S is the same regardless of the location of the unit area in S.



(a) (b)

Figure 1: Convenience stores in Shibuya, Tokyo (a) on a plane with Euclidean distance and (b) on a street-network with the shortest-path distance.

When we analyze spatial phenomena in terms of the distribution of points, these assumptions are not always acceptable. In particular, when we analyze spatial phenomena in a small urban area, these assumptions are hard to accept (Berry, 2002). For instance, consider the marketing analysis of convenience stores in a downtown area (Figure 1b). Consumers go to stores on foot or by car through streets (the arrowed line in Figure 1b), and convenience stores (the points in Figure 1b) are located along streets. Consequently, it is more natural to assume that:

i') space S is represented by a network,

ii') distance in S is measured with the shortest-path distance.

Yet, almost all quantitative methods for spatial analysis assume i) and ii). There are two reasons for adopting these assumptions. First, mathematical treatment on a plane is simpler and more tractable than that on a network. Second, most researchers seem to expect that a network space with the shortest-path distance can be approximated by a plane with Euclidean distance. This expectation, however, is doubtful. An instructive example is shown in Figure 2a. Having seen this distribution, probably nobody believes that the points are uniformly and randomly distributed. But, this is true. Actually, as is seen in Figure 2b, the points (which are the same as those in Figure 2a) are generated according to the uniformly random point process (the binomial point process) on a network. In addition, an empirical study by Yamada and Thill (2002) show that the conclusion obtained from the *K*-function method assuming a plane with Euclidean distance is significantly different from that obtained from the *K*-function method assuming a network with the shortest-path distance.

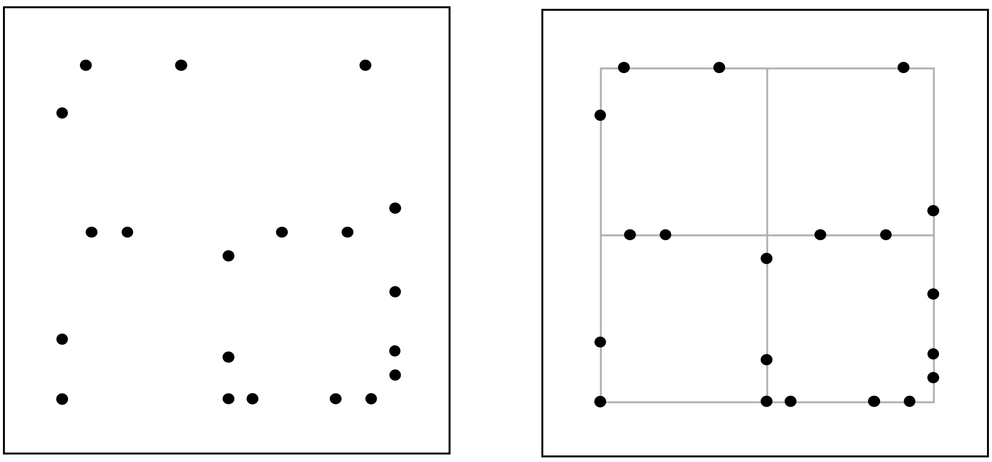


Figure 2: A distribution of points (a) on a plane and (b) on a network (the points are generated by the binomial point process on a network).

To overcome the limitations resulting from Assumptions i) and ii), several methods have been developed under Assumptions i’) and ii’). For example, Okabe *et al*. (1995) formulate the nearest distance method on a network; Okabe and Yamada (2001) and Yamada and Thill (2002) formulate the *K*-function and cross *K*-functions methods on a network; Miller (1994, 1996), Okabe and Kitamura (1996), Morita *et al*. (2001) and Okabe and Okunuki (2001) formulate the Huff model on a network.

In addition to Assumptions i) and ii), the last assumption, Assumption iii), is also problematic in practice. Needless to say, the actual geographical plane is heterogeneous. For example, consider the point process of convenience stores in a downtown. This process may be well examined by considering heterogeneity produced by uneven consumer (population) density. Thus Assumption iii) is hardly acceptable when we examine the actual distribution of convenience stores in a downtown. To overcome this limitation, a few theoretical methods are proposed in the related literature, such as heterogeneous point processes (Diggle, 1983). However, these methods are still hard to deal with actual geographical phenomena.

Fortunately, we have noticed that the above limitations can be overcome if a plane with the Euclidean distance (Assumptions i) and ii)) is replaced with a network with the shortest-path distance (Assumptions i’) and ii’)). The objective of this paper is to show an easy method for dealing with the distribution of points on a ‘heterogeneous network’. Roughly speaking, this method treats a ‘heterogeneous network’ by transforming it into a ‘homogeneous network’. Once this transformation is done, we can apply any methods for spatial analysis on a homogeneous network to spatial analysis on a heterogeneous network. We do not have to develop special methods for a heterogeneous network.

2. The transformation from a heterogeneous network into a homogeneous network

The key concepts are, as often referred to in the above, a 'homogeneous network' and a 'heterogeneous network' . To define them explicitly, let us consider a network, N , consisting of a set, L , of line segments (which may be curved) and a set, P , of nodes, satisfying that the line segments in L are connected (or crossed) only at the nodes in P , and that the end points of the line segments in L are the nodes in P .

For a network N , we define a *homogeneous network* as a network in which the characteristics of a unit line segment in N are the same regardless of the location of the unit line segment in N ; in terms of stochastic point processes, the probability of a point being placed in a unit line segment in N is the same regardless of the location of the unit line segment in N . A *heterogeneous network* is defined as a network that is not a homogenous network. Stated more explicitly, a *heterogeneous network* is defined as a network in which the characteristics of a unit line segment in N vary according to the location of the unit line segment in N ; in terms of stochastic point processes, the probability of a point being placed in a unit line segment in N varies according to the location of the unit line segment in N . The homogeneity or heterogeneity is represented by a density function defined on N . The homogeneous network has the uniform density function, and the heterogeneous network has a non-uniform density function.

In actual analysis, we have to estimate a density function that represents a heterogeneous network. Let us consider how to obtain this density function. Suppose that we want to analyze the distribution of convenience stores in relation to the distribution of households. Owing to recent progress in GIS, we can easily obtain a digital map where stores and houses are represented by polygons, and streets are represented by line segments (Bailey and Gatrell, 1995). First, we obtain a representative point of a house by the center of the polygon representing the house (the white points in Figure 3). Second, we assign the representative point on the nearest point on a network (the black points in Figure 3). We assume this point as the gate of a house facing a street. Third, we count the number of assigned points for each link. Last, we obtain the house (household) density from the number of assigned points on a link divided by the length of the link. This procedure is carried out with the software developed by Okabe *et al.* (2002). Note that in the above method, the density of households may differ from link to link, but it is constant within each link. Alternatively, we may use a variable density within a link. This density function can be estimated by a non-parametric density estimation method using the points on a link obtained in the above procedure (Silverman, 1986).

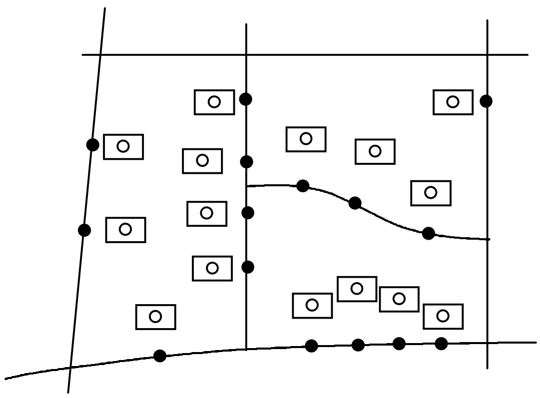


Figure 3: Assignment of houses to the nearest points on a network.

To define a density function for a heterogeneous network mathematically, let L_i $(i = 1, \dots, n)$ be a link in L, and t be the length from an end node of L_i to a point on L_i along L_i. Then the density of an attribute (say, household density) is represented by the function

$$f_i(t), \quad 0 \le t \le t_i, \quad i = 1, \dots, n,$$

where t_i is the length of L_i. Note that the density may be constant over L_i as in the above example (but different from link to link), or it may vary continuously over L_i. We assume that for all i,

$$\int_0^{t_i} f_i(t)dt \ge a > 0,$$

where a is a positive constant. A clue to a method that transforms a heterogeneous network into a homogeneous network is given by a transformation that is often used for generating random variables in statistics (Freund, 1971). To state the transformation explicitly, let x be an arbitrary random variable whose probability density has the value of $f(x)$. We transform the variable x into y as

$$y = \int_{-\infty}^{x} f(x)dx = F(x).$$

This transformation is called the *probability integral transformation* (Figure 4). Let $g(y)$ be the density function of the random variable y. Since the equation

$$\frac{dy}{dx} = \frac{dF(x)}{dx} = f(x)$$

holds, we have

$$g(y) = 1, \quad 0 < y < 1,$$

which is the *uniform probability density function* with the interval $0 < y < 1$. This derivation shows that any non-uniform probability density function is transformed into the uniform probability density function through the probability integral transformation $y = F(x)$. This fact implies that random variables, $x_1, \ldots, x_n$, that follow the non-uniform probability density function $f(x)$ (the black points on the horizontal axis in Figure 4) are equivalent to transformed random variables, $y_1 = F(x_1), \ldots, y_n = F(x_n)$, that follow the uniform probability density function $g(y)$ (the black points on the vertical axis in Figure 4). Conversely, random variables, $y_1, \ldots, y_n$, that follow the uniform probability density function $g(y)$ are equivalent to transformed random variables, $x_1 = F^{-1}(y_1), \ldots, x_n = F^{-1}(y_n)$, that follow the non-uniform probability density function $f(x)$, where $x = F^{-1}(y)$ is the inverse function of $y = F(x)$. Using this property, we can obtain random variables, $x_1, \ldots, x_n$, that follow a non-uniform probability density function $f(x)$ by generating random variables $y_1, \ldots, y_n$ that follow the uniform probability density function $g(y)$ and transforming these random variables through $x_1 = F^{-1}(y_1), \ldots, x_n = F^{-1}(y_n)$.

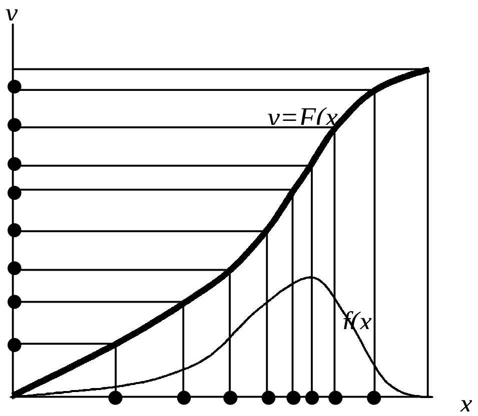


Figure 4: The probability integral transformation.

We can utilize the idea of the probability integral transformation for transforming a heterogeneous network into a homogeneous network. To be explicit, for each link L_i, we define the transformation

$$s = \int_0^t f_i(t)dt = F_i(t), \quad i = 1, \ldots, n.$$

We consider a line L_i^* (that will correspond to L_i), and let s be the length from an end node of L_i^* to a point on L_i^* along L_i^* (Figure 5). We correspond L_i to L_i^* in such a manner that a point at t on L_i corresponds to the point $s = F(t)$ on L_i^*. As a result, the origin node of L_i $(t = 0)$ corresponds to the origin node of L_i^* $(s = F(0) = 0)$, and the other end node of L_i $(t = t_i)$ corresponds to the other end node of L_i^* $(s_i = F(t_i))$.

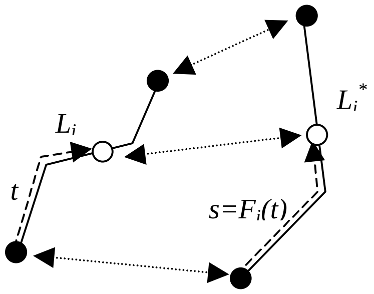
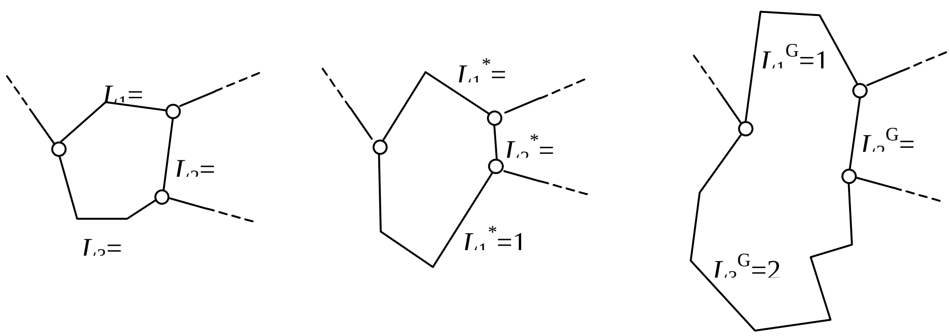


Figure 5: Transforming L_i into L_i^*.

Let L^* be the set of the resulting links $L_1^*,\ldots,L_n^*$, and P^* be the set of nodes generated by the end nodes of the links in L^* (Figure 6b). Then the network N^* consisting of the link set L^* and the node set P^* is the *uniform network* in the sense that the density of a unit line segment on the network is constant regardless of the location of the unit line segment on L^* (Figure 6b). We call the transformation from the network N into the network N^* the *uniform network transformation*.



(a) (b) (c)

Figure 6: Part of (a) N, (b) N^* and (c) N^G (the numbers indicate the length of the links).

We make two important remarks. First, the geometrical forms of the links in L^* are arbitrary. Any form is acceptable as far as they are constructed through the above procedure. Second, the location of a point on a link L_i^* is uniquely determined with the parameter s regardless of the form of L_i^*. In this sense, the form is not essential in the network N^*.

Summing up, we can transform any heterogeneous network into a homogeneous network through the uniform network transformation.

3. Geometrical Realizability of the transformed network

Having found the above nice transformation, we question whether or not the transformed network N^* can be geometrically realized on a plane. The original network N is given on a plane (Figure 6a), but the length of each link in the transformed network N^* changes (Figure 6b), and hence it is questionable that the transformed network N^* is geometrically realized on a two-dimensional space (plane). For example, in Figure 6b, the transformed links might cross the other links. In this section, we examine this question. Note that such a problem never occurs in the probability integral transformation, because the correspondence is between only two lines (the *x*-axis and *y*-axis as in Figure 4).

Let us first define what a 'geometrically realized planar network' of N^* is. Let N^G be a network consisting of a set, L^G, of links and a set of nodes, P^G. For N, N^* and N^G (Figure 6), we consider the following conditions:

i) there exists on-to-one correspondence between links in L^* (or L) and those in L^G,

ii) there exits one-to-one correspondence between nodes in P^* (or P) and those in P^G,

iii) the graph of the network N^G is isomorphic to the planar geometric graph of the network N,

iv) the length of a link L_i^G in L^G is proportional to the length of the corresponding link L_i^* in L^* (the proportion rate is the same for all links $i = 1, \ldots, n$).

When the network N^G satisfies all these conditions, we say that the network is a *geometrically realized planar network* of the transformed network N^*. Fortunately, we can find a geometrically realized planar network N^G of the transformed network N^* through the following procedure.

First, we assume that the configuration of nodes in P^G is the same as that of nodes in P of a given network N (hence (i) is satisfied) (the configuration of the nodes in Figure 6a is the same as that in Figure 6c).

Second, we stretch the length of each link in L^* proportionally until the length of a link in L^* is greater than or equal to the length of corresponding link in L for all links (the property (iv) is kept) (the numbers on the links in Figure 6b are all doubled in Figure 6c to satisfy $L_i \le L_i^G$ for all i).

Third, we obtain a set of polygons $A_1, \ldots, A_n$ that satisfies: the polygon A_i contains the link L_i in L and the polygons do not overlap each other except at end nodes of links in L . An example is shown in Figure 7. Such a set of polygons can be computationally obtained using the decomposed line Voronoi diagram generated by the line segments of L (Okabe *et al*., 2000).

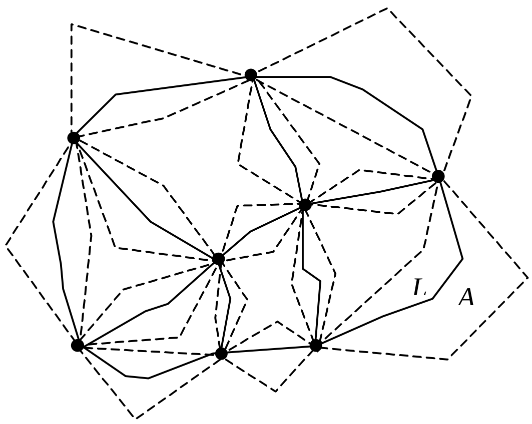
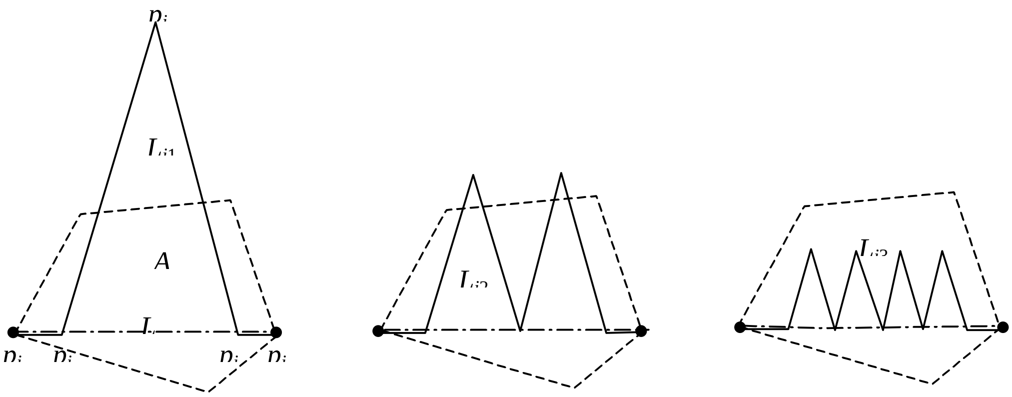


Figure 7: An example of $A_1, \ldots, A_n$

Fourth, for each polygon A_i , we draw a polygonal line L_{i1}^G consisting of points $p_{i1}, p_{i2}, p_{i3}, p_{i4}, p_{i5}$ whose length is equal to the stretched link of L_i^* ((i) is satisfied) (Figure 8a). If the line L_{i1}^G is included in the polygon A_i , let L_i^G be L_{i1}^G , and we stop. If not, we go to the next. Note that for ease of explanation, L_i is assumed to be a straight-line segment, but the following proof can be applied to the case of a polygonal line with slight modifications.



(a) (b) (c)

Figure 8: The procedure for obtaining a geometrically realized planar network

Fifth, we bend the triangle $p_{i2}p_{i3}p_{i4}$ in such a way that the vertex p_{i3} is on the link L_i (see Figure 8b). The length of the bended line, L_{i2}^G, is the same as that of the link L_i^*. If this bended line is included in the polygon A_i, let L_i^G be L_{i2}^G, and we stop. If not, we go to the next.

Sixth, we bend the two triangles in the same manner as in Figure 8c. The length of the bended line, L_{i3}^G, is the same as that of the link L_i^*. If this bended line is included in the polygon A_i, let L_i^G be L_{i3}^G, and we stop. If not, we go to the next.

In this manner, we continue a similar procedure until the bended line, L_{im}^G, is included in the polygon A_i. This procedure always terminates in a finite number, m, of steps. Let L_i^G be L_{im}^G.

When the above procedure terminates, we construct a network N^G with the set of links $L^G = \{L_{1m}^G, \ldots, L_{nm}^G\}$ and the set of nodes $P^G = P$. Now our problem is to examine whether or not the network N^G satisfies the conditions (i)-(iv). Since $P^G = P$, the condition (i) is satisfied. Since there is one-to-one correspondence between L and L^*, and there is one-to-one corresponding between L^* and L^G, the conditions (i) is satisfied. Since the length of a link in L^G is stretched in proportion to the length of the corresponding link in L^*, the condition (iv) is satisfied. Since the graph of the network N is a planar graph and since the link L^G_{im} fits to the corresponding link L_i by continuous deformation without crossing the other links (the polygons $A_1,\ldots,A_n$ do not overlap each other except at nodes), the condition (iii) is satisfied. Therefore the network N^G is a geometrically realized planar network of the uniform network N^* that is transformed from a given heterogeneous network N. This proves that any uniform network that is transformed from a heterogeneous network through the uniform transformation can be geometrically realized on a plane.

4. Conclusions

The major conclusions of this paper are summarized as follows.

i) Any heterogeneous network can be transformed into the uniform (homogeneous) network through the uniform network transformation.

ii) The obtained uniform network can be geometrically realized on a plane.

The implications of these conclusions are very useful. First, we can easily deal with a heterogeneous network. This is a great advantage of spatial analysis on a network, because there exists no transformation from a heterogeneous plane to a homogeneous plane and so spatial analysis on a heterogeneous plane is very hard. Second, any methods for spatial analysis on a homogeneous network can be applied to spatial analysis on a heterogeneous network. Third, we do not have to develop special methods for spatial analysis on a heterogeneous network. We only concentrate on developing methods for spatial analysis on a homogeneous network.

As shown in this paper, spatial analysis on a network has great advantages, but no methods are perfect, and spatial analysis on a network has also some limitations. In practice, computation on a network is usually more time-consuming than that on a plane. This limitation has been greatly diminished by recent progress in computer hardware and processing, but computation on a network still takes much time. Second, in spatial analysis on a network, all geographical features are supposed to be mapped on a network. However, there are some geographical features that cannot be mapped on a network. For example, a large park that is represented by a polygon is difficult to map on a network. Thus spatial analysis on a network is effective only when geographical features can be meaningfully mapped on a network.

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