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A tale of two cities:

Urban spatial structure and mode of transportation

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Abstract

Paying special attention to the interdependence of the spatial structure of a city and the transport mode used there, we obtain two types of equilibria. One is the "auto city equilibrium" at which workers, distributed over the city thinly, use automobile for their commutes. The other is the "rapid transit city equilibrium" at which a transport firm provides rapid transit services for the commutes of workers, who are distributed densely. Examining the conditions for each type of equilibrium, we discuss the possibility of multiple equilibria.

Keywords: auto city; multiple equilibria; population density; rapid transit city; spatial structure of a city, transport firm

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1 Introduction

The spatial distribution of economic activities within a city significantly varies across cities. Notably, some cities extend over a quite broad plain where population and employment spread out fairly evenly, whereas others are compact with a large part of economic activities taking place in a quite small area. This variation is attributable to a number of factors such as histories, social compositions, political environments and the policies toward urban planning. One of the most important is probably a difference in the modes of transportation used in a city. It can be argued, for instance, that the spatial structure of Los Angeles has been shaped based on the use of automobile while that of Paris on the use of horse-drawn wagons, buses, trams or subways.

The question of how the level of transport costs affects the spatial structure of a city is well answered by a standard economic theory on urban structure. The Alonso-Mills-Muth model of land use patterns, for instance, demonstrates that the decline in transport costs leads to a spatial expansion of a city.[1] Indeed, a substantial number of studies argue that spatial decentralization, one of the most pronounced changes that cities have undergone in recent centuries, occurs as a result of the decline in transport costs, most importantly due to the spread of automobile (see Anas et al. (1998), Brueckner (2001), Glaeser and Kahn (2004), and Baum-Snow (2007), among others). In addition, the literature of new economic geography emphasizes the role of transport costs in the determination of the spatial distribution of economic activities: In general, lower transport costs work as a centripetal force (see Fujita et al. (1999) and Combes et al. (2008), among others). However, most researches in the literature do not deal with the spatial distribution *within* a city.

For all that, this explanation tells only one side of a story. The immediate question is why such a divergence in urban structure *persists*. Why did Paris, for example, not switch to an automobile city like today's Los Angels in the latter half of the 20th century? Obvious answer would be that *the mode of transportation used in a city depends on the spatial structure of that city*: In Paris, a highly dense concentration of population and employment along with insufficient road capacities makes the commutes by automobile infeasible. In Los Angels, contrastingly, the provision of rapid transit services is not profitable enough because of relatively thin population and employment in the central business district and/or suburban subcenters (it is only recently that a rapid transit system was constructed there, which carries merely a quite small portion of commuters compared to automobile). The key observation is that *not only is the spatial structure of a city determined by the mode of transportation used there, but also the mode of transportation used in a city is determined by the spatial structure of that city*. In other words, the causality between the spatial structure and the mode of transportation used goes *both way*. We pay attention to this interdependence, which has seldom attracted the concern of researchers, nevertheless.

[1] In that model, however, whether the decline raises population density or not depends on the nature of a city. When it is "closed" (i.e., its population is fixed), it reduces the population density in its inner area. When it is "small and open" (i.e., intercity migration is allowed), to the contrary, it enhances the population density in its entire area.

Acknowledging this interdependence, we can understand the persistence of a certain spatial structure as a result of a lock-in effect of multiple equilibria. That is, in one equilibrium, referred to as an “auto city” equilibrium, population density is low and workers use automobile to go to work. In the other equilibrium, a “rapid transit city” equilibrium, population density is high enough for a transport firm to earn a sufficient amount of revenue to cover the costs of construction and operation of a rapid transit system. Here, a particular urban structure is supported as an equilibrium outcome when a particular mode of transportation is used; but it is no longer so when a different mode is used. At the same time, the use of a particular mode of transportation is supported as an equilibrium outcome when a city has a particular spatial structure; but it is no longer so when a city has a different structure. It is this property that generates the lock-in effect.

The aim of this paper is to examine such a possibility of multiple equilibria through a rigorous analysis. For that purpose, we construct a model of urban land use of the Alonso-Miills-Muth type incorporating a transport firm, and examine in the framework of a two-stage game when each type of equilibrium mentioned above emerges. Then, we analyze the case of multiple equilibria where both types emerge for the same set of parameters. In this case, even if two cities are endowed equally by nature, it may happen that one develops to an auto city like Los Angels and the other to a rapid transit city like Paris by, for instance, a historical accident. This is our tale of two cities.

One significance of this study concerns the recent movement in urban policies toward a more compact city. In many cities, especially those in developed countries, the attempts to concentrate residences and work places in the narrower area of a traditional city center have becoming more and more common. It is expected that they help us take advantage of the benefits of agglomeration economies further and reduce environmental burdens. What is important here is that such attempts often accompany new construction or re-development of a public transportation system represented by subways and trams (LRT’s). We can regard them as the attempts to switch the equilibrium from the auto city equilibrium to the rapid transit city equilibrium. In this context, the newly developed mass transit system probably suffers losses, *given the spatial structure of the auto city*, that is, sparse distributions of population and employment. Therefore, the transit system cannot be run by a private company, and a government needs to continue to pay the losses until the day when the city becomes concentrated enough to yield a sufficient amount of demand for mass transit services. That day may or may not going to come, and even if it comes, it will be in a considerably distant future, because the spatial structure changes only gradually. In this way, our perspective provides an adequate framework to understand and evaluate the attempts toward a compact city.

It is surprising that the relationship of mutual dependence between the spatial structure of a city and the transport mode used there has seldom attracted the concern of researchers. In the field of spatial economics, it is rare that different transport modes are explicitly considered.[2] What

[2]One exception is found in Takahashi (2006), who examines the adoption of a transport technology from “traditional”

is worse, transport costs are usually given exogenously, with noteworthy exceptions of the works by Behrens et al. (2009) and Takahashi (2011), which study how transport costs are endogenously determined and how it affects the spatial distribution of economic activities.

The rest of the paper consists of five sections. In Section 2, the model is presented. We first explain a basic framework, then introduce transportation modes and finally discuss the structure of game. The next section discusses the decision makings of workers and a transport firm. In Section 4, we derive two types of equilibria mentioned above, namely, the auto city equilibrium and the rapid transit city equilibrium, and examine when each type emerges. In Section 5, we explore the two types further by simulation analyses specifying functional forms and parameters. Section 6 concludes.

2 Model

2.1 Basic framework

We consider a linear city with its width being unity in a homogenous plane. Because it is the set of the points within 0.5 miles from the straight line penetrating the middle of it, the city can be interpreted as the area served by a rapid transit line running along that midway line when the maximum walking distance for commuters is 0.5 mile (here, we are ignoring the fact that stations are not continuously but discretely placed). With this interpretation, assuming a linear city is rather more natural than assuming a disk-shaped city as in the standard model as long as only one transit line is concerned.

The city is monocentric, that is, all the production activities are concentrated at the "center" located at its endpoint. Locations within the city is identified by the distance from the city center, $d \geq 0$.

In the city, there are workers who are endowed with 1 unit of time. Spending some portion of it, they work to earn a wage, whose rate is fixed at w. They consume land for housing, a composite good and leisure, whose amounts are denoted by x, z and e, respectively. Here, we take the approach of Train and McFadden (1978) in which workers, facing a trade-off between consuming a greater amount of goods and enjoying longer time for leisure, freely choose the length of leisure time and thus work hours, denoted by n.[3] In order to work, furthermore, workers need to commute to the city center, paying monetary and time costs. We assume that the number of workdays is fixed although the work hours per workday are flexible and chosen by workers. Thus, the costs of commute do not depend on the amounts of leisure consumed by workers, but depend, it is assumed, only on the distance of commute. We denote the monetary and time costs

and "modern" ones, and its effects upon economic geography.

[3]Another approach involves fixed work hours. The length of leisure time is automatically determined as a residual of the total available time after subtracting the work hours and commuting time. The reality is probably in between the worlds depicted by the two approaches. For the differences in their implications, see Jara-Diaz (2007), for instance.

for a worker living at d by $m(d)$ and $t(d)$, respectively, which are increasing functions.

Let us consider a representative worker living at d. Her time constraint is given by $n(d) + e(d) + t(d) = 1$. Furthermore, assuming that the composite good is a numeraire and that land rent is taken by absentee landlords, we obtain their budget constraint as $r(d)x(d) + z(d) + m(d) = wn(d)$, where $r(d)$ is the land rent at d. It is convenient to combine these two constraints into

$$r(d)x(d) + z(d) + we(d) = y(d) \equiv w - c(d), \tag{1}$$

where $c(d) \equiv wt(d) + m(d)$ is a *generalized cost* of commute inclusive of both monetary and time costs. $c(\cdot)$ is an increasing function. Furthermore, $y(d)$, a decreasing function of d, is the income that would be earned by a worker who worked for all the available time, $w[1 - t(d)]$, subtracted by the monetary cost of commute, $m(d)$. That is, it is the measure of a *potential disposable income*.

Workers have the same preference represented by utility function $U(x, z, e)$, which they maximize subject to (1). Because the division of spendings between the two goods and the leisure is not a main concern in this paper, we specify the utility function as $U(x, z, e) = \beta \ln u(x, z) + (1 - \beta) \ln e$. Then, the amount of leisure consumption for the worker at d is

$$e(d) = \frac{(1-\beta)y(d)}{w}. \tag{2}$$

Note that β becomes equal to the ratio of the actually earned disposable income to the potential income, that is,

$$\beta = \frac{wn(d) - m(d)}{y(d)}. \tag{3}$$

Thus, we can interpret β as the relative measure of work hours.

Choosing x and z, workers maximize the sub-utility function, $u(x, z)$, subject to $rx + z = \beta y$. First order necessary condition for the worker at d is given by

$$u_1\Big(x(d),\ \beta y(d) - r(d)x(d)\Big) - r(d)u_2\Big(x(d),\ \beta y(d) - r(d)x(d)\Big) = 0, \tag{4}$$

where $u_i(\cdot)$ denotes a partial derivative of $u(\cdot)$ with respect to its i'th argument.

In a sufficiently long span, the city is small and open, that is, migration occurs between this city and the outside regions so that the level of utility for each worker in the city coincides with that prevailing outside the city, denoted by $\bar{u}$. That is,

$$U\left(x(d),\ \beta y(d) - r(d)x(d),\ \frac{(1-\beta)y(d)}{w}\right) = \bar{u} \tag{5}$$

for any d at a *locational equilibrium*.

Solving (4) and (5) simultaneously, we can obtain the demand for land by each worker and the land rent for each d, which we denote as $\widetilde{x}(d)$ and $\widetilde{r}(d)$ (their arguments may be dropped if doing so causes no confusion). Note that the variables associated with a a particular value of d are determined independently of those associated with a different value of d. In other words, they do not depend on macro variables such as a population distribution and a geographical size of the city. This *independence property*, which is a consequence of the small open city approach, will greatly help us simplify the analysis.

Moreover, outside the city extends agricultural land, which is rented out at a fixed rent, r_a. Then, the location of city's boundary, b, is given by the solution to

$$\widetilde{r}(b) = r_a. \tag{6}$$

The resulting triplet of $\left(\widetilde{x}(d),\ \widetilde{r}(d),\ b\right)$ describes a *spatial structure* of the city.

One qualification needs to be satisfied. The sum of the spendings on land and on the composite good must be nonnegative. Because (1) and (2) imply $r(d)x + z = \beta\left[w - c(d)\right]$, this qualification is expressed as $c(d) \leq w$. We concentrate on the case where r_a is so high and therefore, the city is so small that any $d \in [0, b]$ satisfies this qualification, that is,

$$c(b) \leq w. \tag{7}$$

In this place, it is useful to review the impacts of changes in parameters upon the land consumption because it will play a key role in the determination of the transport mode used in the city. For that purpose, let us define Υ as $\Upsilon \equiv u_{11} - 2u_{12}\widetilde{r} + u_{22}\widetilde{r}^2$ and Ψ as $\Psi \equiv u_{12} - u_{22}\widetilde{r}$, where $u_{ij}(\cdot)$ denotes the second-order partial derivative of $u(\cdot)$ with respect to the i'th and j'th arguments (its arguments are omitted for the clarity). Note that $\Upsilon < 0$ as long as the utility function is strictly quasi-concave and the first order condition, (4), is satisfied. Furthermore, we assume that the land is a normal good, that is, $\partial\widetilde{x}/\partial\left[\beta y(d)\right] \geq 0$. Because differentiating (4) yields $\partial\widetilde{x}/\partial\left[\beta y(d)\right] = -\Psi/\Upsilon$, the assumption is equivalent to

$$\Psi \geq 0. \tag{8}$$

Now, since $\widetilde{x}(d)$ and $\widetilde{r}(d)$ are determined by equations (4) and (5), we can derive the impacts of changes in parameters on $\widetilde{x}(d)$ by totally differentiating them and eliminating a change in $\widetilde{r}(d)$. From (8) and the condition that the marginal rates of substitution become equal to relative prices, that is, $U_x(x,z,e)/U_z(x,z,e) = \widetilde{r}(d)$ and $U_x(x,z,e)/U_e(x,z,e) = \widetilde{r}(d)/w$, the next results follow:

$$\left\{\begin{array}{rcl} \widetilde{x}'(d) & = & -\dfrac{c'(d)}{\Upsilon\widetilde{x}}\left[u_2 + (1-\beta)\Psi\widetilde{x}\right] > 0, \\ \dfrac{\partial\widetilde{x}(d)}{\partial c(d)} & = & -\dfrac{1}{\Upsilon\widetilde{x}}\left[u_2 + (1-\beta)\Psi\widetilde{x}\right] > 0, \\ \dfrac{\partial\widetilde{x}(d)}{\partial\bar{u}} & = & -\dfrac{\widetilde{r}}{\Upsilon\widetilde{x}}\left(u_2 + \Psi\widetilde{x}\right) > 0, \\ \dfrac{\partial\widetilde{x}(d)}{\partial\beta} & = & -\dfrac{\Psi y}{\Upsilon} > 0, \\ \dfrac{\partial\widetilde{x}(d)}{\partial w} & = & \dfrac{1}{\Upsilon w\widetilde{x}}\left[\Big\{\beta y + m(d)\Big\}u_2 + (1-\beta)\Upsilon\widetilde{x}m(d)\right] < 0. \end{array}\right. \tag{9}$$

We can explain these results intuitively. First, *the piece of land demanded by each worker expands as we move farther away from the city center*. As transport costs increase, a disposable income decreases. To accomplish the same utility level, therefore, workers need to increase the consumption of either land or composite good, or both. Second, *the land lot size at a given location shrinks as the transport costs from that location to the city center decline*. The intuition behind this result is the same as before: for the locational equilibrium, it is necessary to allocate a larger land lot to workers who pay higher transport costs. In addition, the finding implies that population density rises as transport costs decline.[4] Third, as the target utility level rises, a worker needs to raise the amount of land consumption. Fourth, the rise in the relative measure of work hours, β, needs to be accompanied with a change that will raise a utility level, namely, the increase in the consumption of land. Finally, the impact of a change in wage rate is twofold. First, its rise brings about an increase in wage income. To keep the utility level constant, therefore, each worker needs to reduce the amount of land consumption other things being equal. This is a standard channel of effect that one can see in the traditional land use theory. What is unique in this paper is, however, that the rise in wage rate implies the increase in the time cost of commute, because a higher value is now attached to a given length of time. As the second line in (9) indicates, this indirect effect is positive. The last line in (9) shows that the former direct effect more than offset the latter indirect effect.

In addition, it is possible to derive by similar analyses the effects of changes in parameters upon the land rent at each location and the location of city boundary. In particular, the effect of a change in transport costs on the land rent is negative, which is relevant for a later discussion:

$$\frac{\partial \widetilde{r}(d)}{\partial c(d)} = -\frac{1}{\widetilde{\widetilde{x}}} < 0. \tag{10}$$

2.2 Transport modes

In the economy, there is a transport "firm", which may be a private firm or a government sector. It constructs and operates a rapid transit system if it decides to do so. To make the analysis simple, we confine our discussion to the case where the firm can construct only one rapid transit line running from the city center to a terminal station located somewhere in the city.

The costs of construction and operation of the rapid transit system depend on the length of the transit line and are expressed by $\Gamma(l)$. The function is increasing and $\Gamma(0) > 0$ because there are some fixed inputs. It is true that in the reality, the costs depend also on the number of passengers, but we disregard it for two reasons. First, the change in variable costs arising from higher ridership is usually much smaller than the change in fixed costs arising from the extension of the line. To a railroad company, in other words, constructing and maintaining infrastructure is a much heavier burden than daily operations. Second, it makes the analysis considerably simple without affecting main results.

[4] One might wonder if this result is inconsistent with the stylized fact that the decline in transport costs, mainly brought about by the use of automobile, is a main reason for the decentralization of a city. This is partly true but partly false. Our model demonstrates that the decline causes not only the increase in population density but also a spatial expansion of a city. Although the former effect may not be observed in the actual decentralization process, the latter effect is indeed one of the most salient features of the process.

In order to make the analysis tractable, we assume that the firm charges for its transport services a price proportional to the distance of a trip. (Although it is not difficult to assume other functional forms for the price, the analysis would become much more complicated without yielding additional insights.) The price of the round trip between the location at d and the city center is $m_R d$, where m_R is determined by the firm.

Workers possibly choose the transport mode for commute from the following two.

First, every worker can drive a car to go to her office. We refer to this mode as mode "auto," or mode A. The time necessary for a commute is proportional to its distance: $t(d) = t_A d$ for $t_A > 0$. The monetary cost is, in contrast, not proportional to the distance but contains a fixed term: $m(d) = m_A d + \mu$ for $m_A > 0$. The fixed term, $\mu > 0$, includes various costs arising from owning an automobile, such as the costs of purchasing a car, most importantly, an insurance fee, a tax payment and a fee for a parking lot. In what follows, we will call $m_A d$ a "variable cost" and μ a "fixed cost" of automobile use, although in the standard theory, these terms are used with respect to the amount of products and not the distance of trips. Furthermore, the money paid by workers for their commutes is taken by someone outside the city, a petroleum industry in Texas, say, or simply burned up as iceberg transport costs, as is often assumed in the literature of new economic geography. Finally, the generalized cost is equal to $c_A(d) \equiv \eta_A d + \mu$ where $\eta_A \equiv m_A + wt_A$.

Second, workers may use a rapid transit, a mode "rapid transit," or mode R. We assume that when a rapid transit is available, the worker living at d can take it from that point. To put it another way, "stations" are placed continuously along the transit line. The time taken for that mode is assumed to be proportional to the distance of a commute: $t(d) = t_R d$. The monetary cost is also proportional to the distance: there is no fixed term for mode R. Since the price of the rapid transit services from d is $m_R d$, $m(d) = m_R d$. The generalized cost is equal to $c_R(d) \equiv \eta_R d$ where $\eta_R \equiv m_R + wt_R$.

2.3 Structure of the game

Analysis is made for the following two-stage game. In the first stage, workers decide whether to live in the city or not; and if they decide to do so, they determine the location of residence and choose the amount of land to consume. At the same time, the transport firm makes a decision on whether to construct and operate a rapid transit system or not. When it decides to enter the industry, it chooses the length of the rapid transit line, l, and the coefficient of its price, m_R. In the second stage, workers decide the transport mode to use.

Recall that the aim of this paper is to discuss the relationship between the spatial structure of a city and the transport mode used there. In this respect, what matters is whether a rapid transit system can be successfully introduced, that is, whether the transport firm can raise a sufficient amount of revenue to cover the necessary costs. However, it will turn out that we can discuss it without going through at what level of price, m_R, and what length of rapid transit line, l, the transport firm actually chooses. Thus, we do not formulate nor solve that part of transport firm's decision problem in the first stage. Therefore, the model is consistent with any pricing scheme, including the average cost pricing that prevails in the world.[5]

3 Decision makings of workers and the transport firm

In this section, we first examine workers' choices in the second stage of the transport mode to use, then, transport firm's decision in the first stage upon whether to enter the industry or not, and finally, workers' decision on the location of residence.

3.1 Workers' choices of transport mode

In the second-stage subgame, the spatial structure of the city has been already determined. Therefore, when the amount of potential disposable income of a worker changes according to her choice of a transport mode, she adjusts the consumptions of composite good and leisure. In other words, a worker who pays $c(d)$ for transport services consumes $\beta\,[w - c(d)] - r(d)x(d)$ units of composite good and $(1-\beta)\,[w - c(d)]\,/w$ units of leisure, where $x(d)$ and $r(d)$ here are the *given* values of the size of land lot and the land rent at each location.

First of all, if the transport firm chose not to enter the industry in the first stage, no rapid transit service is provided and all the workers use automobile. We call such a city an *auto city*.

What we are going to discuss in the rest of this subsection is the case where the transport firm did choose to enter in the first stage, and the city is provided with a rapid transit line up to the terminal at l, which is referred to as an *l-mile rapid transit city*.

Suppose that both mode A and mode R are available for a worker at d. Then, she prefers mode R to mode A if

$$\begin{aligned} U\left(x(d),\ \beta\big[w - c_R(d)\big] - r(d)x(d),\ \frac{(1-\beta)}{w}\big[w - c_R(d)\big] \right) \\ > U\left(x(d),\ \beta\big[w - c_A(d)\big] - r(d)x(d),\ \frac{(1-\beta)}{w}\big[w - c_A(d)\big] \right). \end{aligned} \tag{11}$$

Since the utility function is increasing in each argument, (11) is equivalent to $c_R(d) < c_A(d)$. Furthermore, the worker is indifferent between the two modes if $c_R(d) = c_A(d)$. Here, we assume ad hoc that workers take mode R when indifferent between the two modes.[6] Finally, she prefers mode A to mode R if $c_R(d) > c_A(d)$. By definition, therefore, when both modes are available, a worker at d takes mode R if and only if

$$m_R \le \widehat{m}(d) \equiv \lambda + \frac{\mu}{d}, \tag{12}$$

[5] Our assumption that the production cost does not depend on the number of passengers implies that the rapid transit industry exhibits decreasing costs. It is well-known that some regulations are necessary for decreasing cost industries in order to reduce the loss from natural monopoly. The regulation of average cost pricing is one of such regulations, not only actually observed in a number of cities in the world, but also well justified theoretically: To maximize social welfare under the constraint that a transport firm not suffer losses, the price must be equal to the average cost because the consumer surplus decreases with the price.

[6] This assumption is made only for a technical reason. We can obtain similar results assuming otherwise.

where $\lambda \equiv m_A + w(t_A - t_R)$: $\widehat{m}(d)$ gives the highest price that induces the workers living at d to take mode R. Since $\widehat{m}(\cdot)$ is a decreasing function, however, any $d < d'$ satisfies (12) if d' satisfies it. Consequently, $\widehat{m}(d)$ is actually the highest price that induces all the workers living within d miles from the city center to take mode R. In this sense, we call it an *upper limit price* for d. Moreover, those workers are referred to as *d-mile inner city workers*.

We can view this result from the opposite perspective, using an inverse function, $\widehat{d}(m_R)$:

$$\widehat{d}(m_R) \equiv \begin{cases} \dfrac{\mu}{m_R - \lambda} & \text{if} \quad m_R > \lambda \\ \infty & \text{otherwise.} \end{cases}$$

There are two cases to consider. First, suppose that $m_R > \lambda$. Then, $\widehat{d}(m_R)$ represents the location of the marginal worker who is indifferent between using automobile and using rapid transit, that is, $c_A\big(\widehat{d}(m_R)\big) \equiv c_R\big(\widehat{d}(m_R)\big)$. Thus, the workers living at $d \le \widehat{d}(m_R)$ take mode R if that mode is available at d (also note that (12) is rewritten as $d \le \widehat{d}(m_R)$). This implies that the workers living at $d \le \widehat{D}(l, m_R) \equiv \min\left[\widehat{d}(m_R), l\right]$ take mode R since the rapid transit line is constructed up to the terminal at l (see Figure 1). In contrast, the workers living at $d \in \big(\widehat{d}(m_R), l\big]$, if any, and those living at $d > l$ take mode A because the former do not want to use it even though it is available and the latter have no access to it. Consequently, the $\widehat{D}(l, m_R)$-mile inner city workers take mode R. Second, suppose that $m_R \le \lambda$. In that case, the workers at $d \le l$ have an access to mode R and take that mode because (12) is satisfied. The workers at $d > l$ do not have an access to it and therefore take mode A. For this case, too, we can repeat the conclusion that *the $\widehat{D}(l, m_R)$-mile inner city workers take mode R and the rest take mode A*.

Figure 1: Workers' choices of transport mode

In the above discussion, we have not considered the possibility of "park-and-ride", or the possibility that workers living farther from the city center than the terminal station drive a car from home to a certain station of the rapid transit and then take it to the city center. It turns out, however, that the transport firm can always earn a higher profit by charging the price that does not induce the park-and-ride, as long as the fixed monetary cost of the automobile use, μ, is sufficiently high. Therefore, we do not consider this possibility in this paper. For more detail, see the appendix.

3.2 Entry decision of the transport firm

Let us turn to the decision of the transport firm in the first stage on whether to go into the business or not.

It is now clear from the above discussion on the second-stage subgame that the demand for the transport firm's services comes from the $\widehat{D}(l, m_R)$-mile inner city workers. Since the population density at d is equal to $1/x(d)$, therefore, the revenue of the firm is equal to

$$\Lambda\big(l, m_R;\ x(d)\big) = m_R \int_0^{\widehat{D}(l,m_R)} \frac{d}{x(d)} \mathrm{d}d$$

(the argument $x(d)$ of function $\Lambda(\cdot)$ may be omitted when doing so causes no confusion). The firm enters the industry if and only if its profit, $\Pi(l, m_R) \equiv \Lambda\big(l, m_R;\ x(d)\big) - \Gamma(l)$, is nonnegative.[7]

3.3 Workers' decisions on land consmption

Workers' decision in the first stage on the amount of land to consume has been already explained. To recapitulate briefly, solving (4) and (5) simultaneously yields the size of land lot each worker consumes, $\widetilde{x}(d)$, as well as the land rent at each location, $\widetilde{r}(d)$. The solution further gives the location of a city boundary, b, through (6).

4 Equilibrium

Now we are ready to discuss equilibria. It immediately follows from the above discussion that there are two kinds of equilibria, each of which is associated with a particular type of city, namely, the auto city or the l-mile rapid transit city. We call them an *auto city equilibrium* and a *rapid transit city equilibrium*.

4.1 Auto city equilibrium

In the auto city equilibrium, no rapid transit service is provided and all the workers use automobile. Therefore, $c(d) = c_A(d)$ for any $d \in [0, b^A]$ where b^A is the equilibrium value of b. The size of land lots at each location is determined based on $c(d) = c_A(d)$. It is denoted by $\widetilde{x}^A(d)$, and the corresponding land rent is similarly denoted by $\widetilde{r}^A(d)$.

The necessary and sufficient condition for the auto city equilibrium is that the transport firm suffers a loss by entering the industry. That is, the auto city is supported as an equilibrium outcome if and only if

$$\Lambda^A(l, m_R) < \Gamma(l) \tag{13}$$

for *any* $l \in [0, b^A]$ and *any* $m_R \geq 0$. Here, $\Lambda^A(l, m_R)$ is the value of $\Lambda\big(l, m_R;\ x(d)\big)$ in the auto city, that is, $\Lambda^A(l, m_R) \equiv \Lambda\big(l, m_R;\ \widetilde{x}^A(d)\big)$.

It is obvious that (13) holds for any $l \in [0, b^A]$ and any $m_R \geq 0$ if and only if

$$\max_{l \in [0, b^A],\ m_R \geq 0} \Pi^A(l, m_R) < 0, \tag{14}$$

where $\Pi^A(l, m_R) \equiv \Lambda^A(l, m_R) - \Gamma(l)$. Two observations are important. First, suppose that the price is so high that $\widehat{d}(m_R) < l$. Because the $\widehat{d}(m_R)$-mile inner city workers use the rapid

[7] Here, we assume that the firm enters when the cost equals the revenue. This assumption is arbitrary and can be changed.

transit, the profit increases as the length of the rapid transit line is curtailed from l miles to $\widehat{d}(m_R)$ miles. Second, suppose that the price is so low that $\widehat{d}(m_R) > l$. Then, we can raise the price up to the upper limit price, $\widehat{m}(l)$, without changing the number of users, which yields a higher profit. It follows from these two observations that $m_R = \widehat{m}(l)$ at the maximum of $\Pi^A(l, m_R)$ (for more rigorous derivation, see the proof of the subsequent proposition). Let us define $\Omega^A(l) \equiv \Lambda^A(l, \widehat{m}(l)) = \widehat{m}(l) \int_0^l d/\widetilde{x}^A(d)\,\mathrm{d}d$. We have the following result:

Proposition 1

The auto city is supported as an equilibrium outcome if and only if

$$\Omega^A(l) < \Gamma(l) \tag{15}$$

holds for any $l \in [0, b^A]$.

(For the proof, see the appendix.)

The impact of l on $\Omega^A(l)$ is ambiguous:

$$\Omega^{A\prime}(l) = \frac{\widehat{m}(l)l}{\widetilde{x}^A(l)} - \frac{\mu}{l^2}\int_0^l \frac{d}{\widetilde{x}^A(d)}\mathrm{d}d. \tag{16}$$

The first term of the right hand side represents a marginal effect: As the rapid transit line is extended, additional workers with their mass equal to $1/\widetilde{x}^A(l)$ come to use it, each paying $\widehat{m}(l)l$. The second term, furthermore, shows the effect through the change in price: In order to tempt workers to use a longer transit line, the transport firm needs to lower its price by the amount equal to $|\widehat{m}'(l)| = \mu/l^2$.

It is worthwhile examining the impacts of changes in parameters upon the condition.

First, the auto city becomes more likely to be supported by an equilibrium as the time cost of the use of rapid transit, t_R, rises because

$$\frac{\partial\Omega^A(l)}{\partial t_R} = -w\int_0^l \frac{d}{\widetilde{x}^A(d)}\mathrm{d}d < 0.$$

The rise in t_R, making rapid transit less attractive, lowers the upper limit price, $\widehat{m}(l)$, which reduces the revenue of the transport firm.

Second, the directions of the impacts of changes in the costs of automobile use are ambiguous. As an example, consider the change in m^A:

$$\frac{\partial\Omega^A(l)}{\partial m_A} = \int_0^l \frac{d}{\widetilde{x}^A(d)}\mathrm{d}d - \widehat{m}(l)\int_0^l \frac{d^2}{[\widetilde{x}^A(d)]^2}\,\frac{\partial\widetilde{x}^A(d)}{\partial c_A(d)}\,\mathrm{d}d.$$

For one thing, as a result of the rise in m_A, the upper limit price rises, which directly raises the revenue of the transport firm. This is represented by the first term of the right hand side. At the same time, the higher m_A brings about a more sparse population distribution, which reduces the profitability of rapid transit (remember that $\partial\widetilde{x}^A(d)/\partial c_A(d) > 0$ by (9)). This is captured by the second term. If the latter indirect effect dominates the former direct effect, the overall impact is negative. In that case, the auto city becomes more likely to be supported by an equilibrium as m_A rises. If the indirect effect is dominated by the direct effect, instead, the opposite result holds. The changes in t_A and μ have similar impacts.

Third, recall that the decline of the prevailing utility level ($\bar{u}$) and that of the relative measure of work hours (β) bring about a denser population distribution at each location (see (9)). Therefore, these changes result in a higher revenue of the transport firm, which makes it less likely for the auto city to be supported by an equilibrium.[8]

Fourth and last, the effect of a change in the wage rate (w) is ambiguous. It affects the revenue of the transport firm through two channels. First, it alters a spatial structure: As the wage rate rises, the population density goes up (remember (9)). Second, it changes the upper limit price. Recall that the rise in the wage rate implies the increases in time costs. If the time cost of the use of rapid transit is relatively lower compared to that of automobile, the former mode becomes further attractive as a result of the rise in the wage rate, which raises the upper limit price. If the opposite holds, the upper limit price falls as the wage rate rises. Indeed, we have

$$\frac{\partial \Omega^A(l)}{\partial w} = (t_A - t_R) \int_0^l \frac{d}{\tilde{x}^A(d)}\, \mathrm{d}d - \widehat{m}(l) \int_0^l \frac{d}{[\tilde{x}^A(d)]^2} \frac{\partial \tilde{x}^A(d)}{\partial w}\, \mathrm{d}d.$$

The two terms in the right hand side represent the ambiguous effect through the change in the upper limit price and the positive effect through the change in the spatial structure (recall that $\partial \tilde{x}^A(d)/\partial w < 0$), respectively. As long as the time cost of rapid transit use is not too high compared to that of automobile, the first effect is dominated by the second, and total effect becomes positive.

4.2 Rapid transit city equilibrium

In the rapid transit city equilibrium, rapid transit services are provided. Therefore, $c(d) = c_R(d)$ for $d \in \left[0,\ \widehat{D}(l, m_R)\right]$ and $c(d) = c_A(d)$ for $d \in \left(\widehat{D}(l, m_R),\ b^R\right]$, where b^R denotes the equilibrium value of b. Workers decide the amount of land consumption based on this scheme of transport costs, which is denoted by $\tilde{x}^R(d)$. The corresponding land rent is similarly denoted by $\tilde{r}^R(d)$.

In the first place, two properties of the rapid transit city equilibrium are worth mentioning.

First, when $\widehat{d}(m_R) \leq l$, the amount of the land consumed by each worker and the land rent are continuous at $d = \widehat{D}(l, m_R)$ since $c_A\big(\widehat{d}(m_R)\big) = c_R\big(\widehat{d}(m_R)\big)$. They, however, have a kink at that point, that is, their derivatives with respect to the distance exhibit discontinuity. When $\widehat{d}(m_R) < l$, in contrast, they are discontinuous at $d = \widehat{D}(l, m_R)$.

Second, the variables at the suburban part of the rapid transit city where workers use automobile coincide with those at the same location in the auto city. In other words, $\tilde{x}^R(d) = \tilde{x}^A(d)$ and $\tilde{r}^R(d) = \tilde{r}^A(d)$ for any $d > \widehat{D}(l, m_R)$. It is a consequence of the independence property mentioned

[8] For $\chi \in \{\bar{u}, \beta\}$, (9) implies that $\dfrac{\partial \Omega^A(l)}{\partial \chi} = -\widehat{m}(l) \displaystyle\int_0^l \frac{d}{[\tilde{x}^A(d)]^2} \frac{\partial \tilde{x}^A(d)}{\partial \chi}\, \mathrm{d}d < 0.$

earlier. One implication is that the locations of the boundaries of the two types of cities coincide with each other, $b^R = b^A$. Another implication is that the city population is greater at the rapid transit city equilibrium than at the auto city equilibrium. The reason is as follows: Recall that the size of land consumed by each worker increases with transport costs (see the second equation in (9)). Therefore, $\tilde{x}^R(d) < \tilde{x}^A(d)$ for $d \in \left[0,\ \widehat{D}(l, m_R)\right)$ since $c_R(d) < c_A(d)$ for $d < \widehat{d}(m_R)$. Furthermore, $\tilde{x}^R(d) \leq \tilde{x}^A(d)$ for $d = \widehat{D}(l, m_R)$ since $c_R(d) \leq c_A(d)$ for $d \leq \widehat{d}(m_R)$. Therefore, the population of the interval $d \in \left[0,\ \widehat{D}(l, m_R)\right]$ in the rapid transit city is greater than that in the auto city. The populations of the interval outside of it, i.e., $d \in \left(\widehat{D}(l, m_R),\ b\right]$ are equal for the two types of cities because of the property derived above.

Now, let us consider the conditions for the rapid transit city equilibrium. It immediately follows from the analysis of transport firm's entry decision that the l-mile rapid transit city is supported as an equilibrium outcome if there exists $m^* \geq 0$ such that

$$\Lambda^R(l,\ m^*) \geq \Gamma(l). \tag{17}$$

Here, $\Lambda^R(l, m_R)$ is the value of $\Lambda\big(l, m_R;\ x(d)\big)$ at the rapid transit city equilibirum, that is, $\Lambda^R(l, m_R) \equiv \Lambda\big(l, m_R;\ \tilde{x}^R(d)\big)$. Remember that we do not discuss the choice of a price by the transport firm. It means that we are not to obtain m^* that is "optimal" from a certain viewpoint (i.e. maximizing profit) for the transport firm. Furthermore, note that there are two channels through which the price affects the revenue. One is a direct channel, which involves a change in the payment by each worker, and the other is an indirect channel, which involves a change in the spatial structure of the city. For the latter channel, one caution is necessary. The price does not "affect" the spatial structure in the sense of usual chronological causality: The spatial structure and the price are determined simultaneously in our game. What is intended here is that the equilibrium spatial structure and the equilibrium price are mutually dependent on each other.

It is obvious that there exists $m^* \geq 0$ that satisfies (17) if and only if

$$\Omega^R(l) \equiv \max_{m_R \geq 0}\ \Lambda^R(l, m_R) \geq \Gamma(l), \tag{18}$$

which establishes the following result:

Proposition 2
The l-mile rapid transit city is supported as an equilibrium outcome if and only if (18) holds.

It is straightforward to see $\Omega^{R\prime}(l) \geq 0$ for almost all l, because the envelope theorem implies that it is equal to $m_R l / \tilde{x}^R(l)$ if $l < \widehat{d}(m_R)$ and 0 if $l > \widehat{d}(m_R)$.[9]

Two observations follow.

First, the solution to the maximization problem in (18) can be lower than the upper limit price, $\widehat{m}(l)$, that is, $\widehat{d}(m_R)$ can exceed l at the solution. This exhibits a sharp contrast to the finding for the

[9] $\Omega^{R\prime}(l)$ is not defined at $l = \widehat{d}(m_R)$.

auto city equilibrium that raising the price up to the upper limit price results in the increase in the revenue. The reason is that the change in the price invokes a change in the spatial structure of the rapid transit city to be determined simultaneously. To see this, suppose that the price falls from the upper limit price. Because the users of rapid transit are still the l-mile inner city workers, the revenue declines *other things being equal*. However, the other things are not equal: Corresponding to the fall in the price, the population distribution realized becomes denser, which gives a positive impact on the revenue.

Second, the solution to the maximization problem can be greater than the upper limit price. This finding is also different from that obtained for the auto city equilibrium. Here, our concern is to maximize the revenue for a *given* length of the rapid transit line. Consequently, there is a possibility that a part of the line is left unused at the maximum.

These two observations are easily verified by computing the following derivative:

$$\frac{\mathrm{d}\Lambda^R(l, m_R)}{\mathrm{d}m_R} = \int_0^{\widehat{D}} \frac{d}{\tilde{x}^R(d)} \cdot \left[1 - \frac{\partial \ln \tilde{x}^R(d)}{\partial \ln m_R}\right] \mathrm{d}d \; + \; \frac{\widehat{D}^2}{\tilde{x}^R(d)} \cdot \frac{\partial \ln \widehat{D}}{\partial \ln m_R} \tag{19}$$

(for the sake of clarity, the arguments of $\widehat{d}(m_R)$ and $\widehat{D}(l, m_R)$ are dropped in some cases). The right hand side shows three components of the effects of the rise in a price. The first, represented by the first term in the square brackets ("1"), is a positive direct effect. The second component, captured by the second term in the brackets, is a negative effect of the decrease in the population density. The last component, appearing at the end, is a nonpositive effect that the range of the locations of users is shortened. The first observation above concerns the case with $m_R < \widehat{m}(l)$, or equivalently, $l < \widehat{d}(m_R)$. In that case, $\widehat{D}(l, m_R) = l$ and therefore, the last term of (19) disappears. The overall derivative can be negative because we have the negative density effect. If it is negative, the maximum may be attained at $m_R < \widehat{m}(l)$. Instead, the second observation above concerns the case with $m_R > \widehat{m}(l)$, or equivalently, $l > \widehat{d}(m_R)$. In this case, $\widehat{D}(l, m_R) = \widehat{d}(m_R)$ and therefore, the negative last term remains. Nonetheless, the overall derivative can be positive. Then, $m_R > \widehat{m}(l)$, or equivalently, $l > \widehat{d}(m_R)$ can give the maximum.

Furthermore, it is straightforward to examine the impacts of changes in parameters upon $\Omega^R(l)$.

First, $\Omega^R(l)$ increases or remains unchanged as the costs of automobile use rise, that is, as m_A, μ and/or t_A rises. This is because the change is transmitted only through $\widehat{d}(m_R)$, and

$$\frac{\partial \Omega^R(l)}{\partial \widehat{d}} = \begin{cases} \dfrac{m_R}{\tilde{x}^R(\widehat{d})} > 0 & \text{if} \quad \widehat{d}(m_R) < l \\ 0 & \text{if} \quad \widehat{d}(m_R) > l. \end{cases}$$

(The derivatives of $\Omega^R(l)$ are not defined at $\widehat{d}(m_R) = l$ because $\Gamma^R(l, m_R)$ has a kink at that point.)

Second, $\Omega^R(l)$ decreases as the time cost of rapid transit use, t_R, rises. For this change, we obtain

$$\frac{\partial \Omega^R(l)}{\partial t_R} = \begin{cases} -w\left[\dfrac{\mu m_R}{(m_R-\lambda)^2 \tilde{x}^R(\widehat{d})} + \displaystyle\int_0^{\widehat{d}} \left\{\dfrac{d}{\tilde{x}^R(\widehat{d})}\right\}^2 \dfrac{\partial \tilde{x}^R(\widehat{d})}{\partial c_R(d)} \mathrm{d}d\right] < 0 & \text{if} \quad \widehat{d}(m_R) < l \\ -w m_R \displaystyle\int_0^l \dfrac{d}{\tilde{x}^R(\widehat{d})} \dfrac{\partial \tilde{x}^R(\widehat{d})}{\partial c_R(d)} \mathrm{d}d < 0 & \text{if} \quad \widehat{d}(m_R) > l. \end{cases}$$

The first term of the right hand side in the first line represents the effect of the fall in the number of users ($\widehat{d}$). The second term captures the effect of the change in the spatial structure toward a lower population density (recall that we have seen in (9) that $\partial \tilde{x}^R(\widehat{d})/\partial c_R(d) > 0$).

Third, the changes that lead to a denser population distribution, namely, the decrease in the prevailing utility level and that in the relative work hours, give a favorable effect on the rapid transit city equilibrium through a rise in $\Omega^R(l)$.[10]

Fourth and finally, the effect of a change in the wage rate is ambiguous.

$$\frac{\partial \Omega^R(l)}{\partial w} = \begin{cases} m_R\left[\dfrac{\mu \widehat{d}(t_A - t_R)}{(m_R-\lambda)^2 \tilde{x}^R(\widehat{d})} - \displaystyle\int_0^{\widehat{d}} \left\{\dfrac{d}{\tilde{x}^R(\widehat{d})}\right\}^2 \dfrac{\partial \tilde{x}^R(\widehat{d})}{\partial w} \mathrm{d}d\right] & \text{if} \quad \widehat{d}(m_R) < l \\ -m_R \displaystyle\int_0^l \dfrac{d}{\tilde{x}^R(\widehat{d})} \dfrac{\partial \tilde{x}_R(\widehat{d})}{\partial w} \mathrm{d}d > 0 & \text{if} \quad \widehat{d}(m_R) > l. \end{cases}$$

The two terms of the right hand side in the first line represent the ambiguous effect of the change in the number of users and the positive effect of the change in the spatial structure (recall that $\partial \tilde{x}^R(d)/\partial w < 0$), respectively. As long as the time cost of rapid transit use is not too high compared to that of automobile use, the first effect is dominated by the second, and the total effect becomes positive.

4.3 Discussions

In this subsection, we answer two questions related to the two types of equilibria obtained above: One is the question of when multiple equilibria emerge, and the other is that of which type of equilibrium yields a higher welfare level.

4.3.1 Multiple equilibria

One of the interesting consequences of our model is a possibility that *both* the auto city equilibrium and the rapid transit city equilibrium are realized for the same set of parameters. In this case of multiple equilibria, an identical city becomes the auto city in some circumstances and the rapid transit city in others, depending on the factors outside economic considerations such as historical accidents and expectations.

A necessary and sufficient condition for the multiple equilibria immediately follows from Proposition 1 and Proposition 2:

[10] Since $\partial \Omega^R(l)/\partial \chi = \partial \Lambda^R(l, m_R)/\partial \chi$ for $\chi = \{\bar{u}, \beta\}$, (9) implies that $\partial \Omega^R(l)/\partial \bar{u} = \Psi^{\bar{u}} < 0$ and $\partial \Omega^R(l)/\partial \beta = \Psi^{\beta} < 0$, where $\Psi^{\chi} \equiv -m_R \displaystyle\int_0^{\widehat{d}} \frac{d}{[\tilde{x}^R(d)]^2} \cdot \frac{\partial \tilde{x}^R(d)}{\partial \chi} \mathrm{d}d$.

Corollary 1

Both the auto city and the l^*-mile rapid transit city are supported as equilibrium outcomes for the same parameter set if both the following conditions are met: (15) is satisfied for any $l \in [0, b^A]$, and

$$\Omega^R(l^*) \geq \Gamma(l^*) \tag{20}$$

holds.

We have seen above that the directions of the effects of a change in a parameter on $\Omega^A(l)$ and $\Omega^R(l)$ may be ambiguous. Even if they are unambiguously determined, however, the effect on $\Omega^A(l)$ and that on $\Omega^R(l)$ go in the same direction as in the case of changes in t_R, $\bar{u}$ and β. Therefore, it is hard to decide analytically the directions of the impacts of changes in parameters on the multiple equilibria.

4.3.2 Comparison of welfare levels

The next task is to compare the levels of social welfare realized at the two types of equilibria.

Recall that the utility level of each consumer is pegged at a fixed level at the equilibria because we are considering a small open city. What we need to take into account for the computation of social welfare are, therefore, the land rents earned by absentee landlords and the profit of the transport firm when rapid transit is introduced.

First, when we consider the welfare of not only workers in the city but also that of the landlords who live outside it, it is necessary to count land rents as a part of social welfare. Here, remember the independence property: The land rent in the rapid transit city at any location farther away from the city center than the terminal station coincides the counterpart in the auto city. Therefore, the difference in the total land rents in the two types of equilibria is equal to $\int_0^{\widehat{D}} \widetilde{r}^R(d) - \widetilde{r}^A(d) \, \mathrm{d}d$. However, we know that the land rent is negatively related to the transport costs (see (10)). Consequently, $\widetilde{r}^R(d) > \widetilde{r}^A(d)$ for $d \in [0, \widehat{D}(l, m_R))$ since $c_R(d) < c_A(d)$ for $d < \widehat{d}(m_R)$, whereas $\widetilde{r}^R(d) \geq \widetilde{r}^A(d)$ for $d = \widehat{D}(l, m_R)$ since $c_R(d) \leq c_A(d)$ for $d \leq \widehat{d}(m_R)$. These observations imply that $\int_0^{\widehat{D}} \widetilde{r}^R(d) - \widetilde{r}^A(d) \, \mathrm{d}d \geq 0$: Social welfare is higher at the rapid transit city equilibrium than at the auto city equilibrium when one takes into account the absentee landlords.

Second, when we consider both the land rents earned by absentee landlords and the profit of the transport firm, social welfare is still higher at the rapid transit city equilibrium than at the auto city equilibrium, because the profit cannot be negative.

Thus, we have established the following result:

Proposition 3

Social welfare is higher at the rapid transit city equilibrium than at the auto city equilibrium.

5 Numerical simulations

The key functions, $\Omega^A(l)$ and $\Omega^R(l)$, are too complicated to characterize the two types of equilibria analytically. In this section, we, specifying functional forms and parameters, explore them through numerical simulations.

5.1 Specification of functional forms and parameters

5.1.1 Utility function and cost function

Suppose that the preference over the consumptions of land and composite good is represented by a log linear utility function, $u(x,z) \equiv \alpha \ln x + (1-\alpha)\ln z$. The demand for land and the land rent that simultaneously solve (4) and(5) become

$$\widetilde{x}(d) = \frac{\theta}{\widetilde{r}(0)} \cdot \left[\frac{w-c(0)}{\{w-c(d)\}^{1-\theta}}\right]^{\frac{1}{\theta}} \quad \text{and} \quad \widetilde{r}(d) = \widetilde{r}(0)\left[\frac{w-c(d)}{w-c(0)}\right]^{\frac{1}{\theta}},$$

respectively, where $\theta \equiv \alpha\beta$ and $\widetilde{r}(0) \equiv \theta\big[(1-\alpha)\beta\big]^{\frac{1-\alpha}{\alpha}}(1-\beta)^{\frac{1-\beta}{\theta}}\bar{u}^{-\frac{1}{\theta}}w^{-\frac{1-\beta}{\theta}}[w-c(0)]^{\frac{1}{\theta}}$ are positive constants. Moreover, we can derive from (6) the location of the city boundary as a solution to

$$c(b) = w - \big[w-c(0)\big]\left[\frac{r_a}{\widetilde{r}(0)}\right]^{\theta},$$

which implies that the boundary qualification (7) is always satisfied for this preference.

For the sake of exposition, furthermore, we focus on the benchmark case of linear $\Gamma(l)$, i,e., $\Gamma(l) = \bar{\gamma} + \gamma l$ for some $\bar{\gamma} > 0$ and $\gamma > 0$, which are respectively referred to as a "fixed cost" and a "marginal cost" not with respect to the amount of products but with respect to the length of the rapid transit line.

In the rest of the paper, we evaluate the revenue of the transport firm in terms of the total land rent at the city center in the auto city, i.e., $\widetilde{r}^A(0)$:

$$\begin{aligned}
\frac{\Lambda^i(l,m_R)}{\widetilde{r}^A(0)} &= \frac{m_R}{\theta\,[w-c_A(0)]^{\frac{1}{\theta}}}\int_0^{\widehat{D}} d\cdot[w-c_i(d)]^{\frac{1-\theta}{\theta}}\,\mathrm{d}d \\
&= \begin{cases} \dfrac{m_R}{\eta_A^2(1+\theta)(w-\mu)^{\frac{1}{\theta}}}\cdot\left[\theta(w-\mu)^{\frac{1+\theta}{\theta}} - (w-\eta_A\widehat{D}-\mu)^{\frac{1}{\theta}}\{\theta(w-\mu)+\eta_A\widehat{D}\}\right] & \text{for} \quad i = A, \\ \dfrac{m_R}{\eta_R^2(1+\theta)(w-\mu)^{\frac{1}{\theta}}}\cdot\left[\theta w^{\frac{1+\theta}{\theta}} - (w-\eta_R\widehat{D})^{\frac{1}{\theta}}(\theta w+\eta_R\widehat{D})\right] & \text{for} \quad i = R. \end{cases}
\end{aligned}$$

Here, the assumption that the transport costs are linear in d, that is, $c_A(d) = \eta_A d + \mu$ with $\eta_A \equiv m_A + wt_A$ and $c_R(d) = \eta_R d$ with $\eta_R \equiv m_R + wt_R$, enables us to calculate the integral. Furthermore, the boundary qualification (7), which we have shown to be satisfied, guarantees that $w - c_i(d)$ in the integral is greater than 0 for $i \in \{A, R\}$.

5.1.2 Parameters

For the time cost coefficients, Brueckner's model case furnishes a clue (Brueckner (2001)). He considers a worker who commutes 250 times a year by driving a car at 30 miles per hour.[11] However, one may argue that the driving speed of 30 miles per hour is rather too high in many cities in the world: In most of European and Japanese big cities, for example, the average driving speed is much lower. Thus, we use the speed of 25 miles per hour instead of 30 miles. Computing the coefficient for such a worker, we obtain $t_A = \widetilde{t}_A \equiv 0.002283$.[12] Furthermore, we use $\widetilde{t}_R \equiv 0.8\, \widetilde{t}_A$ as a benchmark value.

For the monetary costs of automobile use, furthermore, we can use the estimates by the American Automobile Association (AAA) and the Automobile Association in the United Kingdom (AA).[13] The AAA reports that the variable costs are equal to \$0.1454 for a small sedan and \$0.1718 for a medium sedan, both per vehicle-mile.[14] The figures of the AA are much higher: \$0.3466 for a car whose price ranges from £13,000 to £18,000, and \$0.3743 for a car whose price ranges from £18,000 to £25,000, both per vehicle-mile. In this paper, we adopt a figure in between those in the two countries, \$0.3.[15] Then, assuming that a typical worker works 250 days a year, we can obtain the parameter for the yearly variable costs as $m_A = \widetilde{m}_A \equiv 150$ $(= 0.3 \cdot 2 \cdot 250)$. For the fixed cost, the AAA gives \$4,548 for a small sedan and \$6,139 for a large sedan, both per year. The figures in AA's are not too different: \$4,936 for a car from £13,000 to £18,000, and \$6,030 for a car from £18,000 to £25,000, both per year. Here, we take \$5,000, that is, $\mu = \widetilde{\mu} \equiv 5000$. In the following simulations, benchmark figures of $t_A = \widetilde{t}_A$, $m_A = \widetilde{m}_A$, $\mu = \widetilde{\mu}$ and $t_R = \widetilde{t}_R$ are used unless otherwise mentioned.

In the United States, household median income is equal to \$53,657 (the US Census) and the

[11]The figure of 250 for the yearly work days is not far from that indicated by Japanese data (*Monthly Labor Survey*), which is 248.4 in 2000, 253.2 in 2005, and 228 in 2010.

[12]The worker spends $250 \cdot 2/25 = 20$ hours on a 1-mile commute (2 miles for a round trip) in one year, which occupies $20/(365 \cdot 24) = 0.002283$ of all the available time.

[13]Several economists have also estimated the costs. Small and Verhoef (2007) estimate that the private cost of operation and maintenance, the vehicle capital cost, roadway cost and parking cost are \$0.141, \$0.170, \$0.016 and \$0.007 per vehicle-mile, respectively, in the United States, which totals \$0.334. Moreover, Roy (2000) cites figures of €0.462 and €0.449 per vehicle-km (or, equivalently \$0.752 and \$0.731 per vehicle-mile) at peak hours and off-peak hours, respectively, for a small gasoline car in the urban areas of the United Kingdom excluding London area. Also he estimates the cost at €0.497 and €0.310 per vehicle-km (or, equivalently \$0.810 and \$0.505 per vehicle-mile) for a small gasoline car and diesel car, respectively, in the urban areas of France. Here, the exchange rate of \$1 = €0.987 in January of the year 2000, which is taken from Shams (2005), is used.

[14]The figures presented here are those in 2015 for the AAA and those in 2014 for the AA. They are obtainable at their web sites. Furthermore, the variable costs in this paper correspond to the "operating costs" in AAA's report and the "running costs" in AA's, whereas the fixed costs correspond to the "ownership costs" in AAA's and the "standing charge" in AA's. Their definitions are, however, quite similar. The variable costs include the costs of fuel, tires and maintenance; and the fixed costs include insurance, taxes, depreciation and finance charge. In addition, AA's original data are in terms of pounds, which are converted to the American dollars based on the exchange rate in 2015 provided by the IMF.

[15]Here, we take into account the fact that the price of gasoline is exceptionally low in the United States compared to those in other advanced countries.

average annual hours actually worked per worker are 1,789 (the OECD), which occupies 20.42% of all the time, both in 2014. From these numbers, we estimate $w = 53,657/0.2042 = 262,767$ (which is the amount of money that one could earn in one year by working 24 hours a day).

Finally, we estimate preference parameters. First, the share of spending on housing is usually 10% to 20%.[16] Thus, we assume that the share of spending on land, α, is equal to 15%. Second, for β, we evaluate (3) for an "average" worker. Our computation on a hypothetical average worker suggests that the reasonable size of β is 18% to 21%; and thus we assume that $\beta = 0.2$.[17] Lastly, $\alpha = 0.15$ and $\beta = 0.2$ yield $\theta = 0.03$.

Now, we are ready to conduct numerical simulations based on these benchmark values of parameters.

5.2 Simulation analysis of the two types of equilibria

To begin with, let us examine function $\Omega^A(l)$. Figure 2 depicts it as an Ω^A *revenue curve*. Each of the five curves corresponds to a different value of t_A. It is apparent that for sufficiently small values of l, the curve has an upward slope. In the benchmark case with $t_A = \widetilde{t}_A$, for instance, it is upward sloping for $l \in [0,\ 39.43)$. For such values of l, the positive effect of the extension of the rapid transit line, which arises from the increase in the number of users (the first term of the right hand side in (16)), dominates the negative effect due to the decline in a price (the second term).

Figure 2: Ω^A revenue curve

Next, let us turn our attention to function $\Omega^R(l)$. We have seen that the impact of a change in m_R on the revenue of the transport firm at the rapid transit city, $\Lambda^R(l, m_R)$, may be positive or negative, and therefore, there is a possibility that the maximum of $\Lambda^R(l, m_R)$ is attained at $m_R \neq \widehat{m}(l)$. However, simulation analyses show that such a case is not likely to occur for plausible values of parameters. In other words, it is likely that $\mathrm{d}\Lambda^R(l, m_R)/\mathrm{d}m_R$ is positive for $m_R < \widehat{m}(l)$ and negative for $m_R > \widehat{m}(l)$, which implies that the maximum is attained at $m_R = \widehat{m}(l)$, the upper limit price for l.

To see this, we can take a look at Figure 3, which describes the relationship between m_R and $\Lambda^R(l, m_R)$. The upward sloping curves represent the revenues for $m_R < \widehat{m}(l)$. Because $\widehat{D}(l, m_R)$ is equal to l in this case, the revenue depends on l and thus, each curve corresponds to a particular

[16] In Japan, for example, it is 16.2% for owner-occupied housing and 13.5% for rented housing in 2014 (*Family Income and Expenditure Survey*).

[17] Suppose that the average worker lives at 10 miles away from the city center and commutes by a car. We can obtain the estimates of $m(10)$ and $t(10)$ using $t = \widetilde{t}_A$, $m = \widetilde{m}_A$ and $\mu = \widetilde{\mu}$. Furthermore, wn is estimated at \$53,657, the household median income. Using these figures, we obtain $\beta = 0.1884$ from (3). In some countries, a rapid transit is widely used for commutes. If the marginal cost parameters for the rapid transit use are the same for the automobile use, that is, $t_R = \widetilde{t}_A$ and $m_R = \widetilde{m}_A$, β is equal to 0.20432. The estimate is not sensitive to the changes in cost parameters: If $t_R = 0.5\widetilde{t}_A$ and $m_R = 0.5\widetilde{m}_A$, the corresponding β only slightly changes to 0.20485, for instance.

value of l. As l rises, the revenue increases for given m_R and the curve shifts upward. Furthermore, the dashed curve represents the revenue for $m_R > \widehat{m}(l)$. For such m_R, $\widehat{D}(l, m_R)$ is equal to $\widehat{d}(m_R)$ and does not depend on l. This curve intersects each of the upward sloping curve at $\widehat{m}(l)$. For instance, the intersection, point A, of that curve and the upward-sloping curve for $l = 15$ is at $\widehat{m}(15)$. Putting these two kinds of curves altogether, the revenue is described by a mountain-shaped kinked curve with its summit at $\widehat{m}(l)$. Again for $l = 15$, for example, it is described by curve OAB. As long as the dashed curve is downward sloping at $\widehat{m}(l)$, the maximum revenue is attained at that point. For the parameters we are considering, it is the case for $l \leq 61.6$, which is likely to hold in a real city. In the rest of the paper, therefore, we concentrate on this case with

$$\Omega^R(l) = \Lambda^R\big(l,\ \widehat{m}(l)\big). \tag{21}$$

Figure 3: The relationship between the price and the revenue in the rapid transit city

When (21) is true, we have

$$\Omega^R(l) - \Omega^A(l) = \widehat{m}(l) \int_0^l d \left[\frac{1}{\widetilde{x}^R} - \frac{1}{\widetilde{x}^A} \right] \mathrm{d}d.$$

That is, the difference between the values of Ω functions for the two types of equilibria depends only on the difference in population densities. Furthermore, note that for $m_R = \widehat{m}(l)$, $c_R(d) - c_A(d)$ is equal to $\mu(d-l)/l$, which is negative for $d < l$. Because higher $c(d)$ implies higher $\widetilde{x}(d)$ (see (9)), it follows that $\Omega^R(l) - \Omega^A(l) > 0$. In other words, the lower transport cost at the rapid transit city than at the auto city brings about a denser spatial structure, which yields a higher revenue. This result implies that for any $l \geq 0$, there exists some $\Gamma(l)$ that satisfies both (15) and (20), that is, there is always a possibility of multiple equilibria.

Figure 4 depicts $\Omega^R(l)$ as an Ω^R *revenue curve* as well as the Ω^A revenue curve. Also described are five dashed straight lines, which represent $\Gamma(l)$'s. We call them *cost curves*. Their intercepts with the vertical axis are equal to $\bar{\gamma}$'s, the fixed costs with respect to the length of the transit line, and their slope represents γ, the marginal cost (in the figure, $\gamma = 0.19$).

Figure 4: The revenue curves and the cost curve

The figure illustrates how different sets of equilibria emerge depending on parameters, which is the crux of the paper. Consider the situation where the fixed cost, $\bar{\gamma}$, gradually declines.

First, when it is sufficiently high that the corresponding cost curve is given by $\Gamma_1(l)$ in Figure 4, the auto city is supported by an equilibrium because the cost curve lies above the Ω^A revenue curve for the entire range of l (see Proposition 1). At the same time, no rapid transit city can be supported by an equilibrium because the cost curve is situated also above the Ω^R revenue curve for the entire range of l (see Proposition 2).

Second, as the fixed cost gradually declines, the cost curve finally touches the Ω^R revenue curve. In the figure, it occurs when $\bar{\gamma} = \bar{\gamma}^{Ru} = 1.911$. After it reaches that critical value, the rapid transit city is supported as an equilibrium outcome for some l. For instance, when the cost curve shifts to $\Gamma_2(l)$, any rapid transit city where the transit line is constructed up to the terminal located in between l_3 and l_5 is supported by an equilibrium. Thus, the critical value $\bar{\gamma}^{Ru}$ represents *the upper limit of the fixed cost for which the rapid transit city is supported by an equilibrium*. Note that the auto city is still an equilibrium outcome because the cost curve remains to lie over the Ω^A revenue curve in the entire range. In this case with a medium size fixed cost, therefore, there are multiple equilibria (see Corollary 1).

Third and finally, when the fixed cost declines to another critical value $\bar{\gamma}^{Al} = 0.823$, the cost curve becomes tangent to the Ω^A revenue curve. Afterward, it becomes possible that we can introduce rapid transit successfully into the auto city and therefore, that type of city is no longer supported as an equilibrium outcome. When the cost curve is described by $\Gamma_3(l)$, for example, the rapid transit line with length between l_2 and l_4 is feasible in the auto city. The critical value $\bar{\gamma}^{Al}$, thus, represents *the lower limit of the fixed cost for which the auto city is supported by an equilibrium*. At the same time, a rapid transit city continues to be supported by an equilibrium. Indeed, for the cost curve given by $\Gamma_3(l)$, any rapid transit city with the transit line extending to the terminal between l_1 and l_6 is supported as an equilibrium outcome.

The two critical values of the fixed cost, $\bar{\gamma}^{Ru}$ and $\bar{\gamma}^{Al}$, which summarize the equilibrium patterns, are useful in discussing the effects of changes in parameters. Here, because of the limitation of space, we discuss only one example, namely, the changes in the costs of automobile use. It is often observed that governments encourage or discourage the use of that mode by controlling the amounts of monetary costs through various policy instruments such as diverse kinds of taxes and subsidies. How do such policies affect the likelihood that the auto city and the rapid transit city emerge? Furthermore, do they widen the possibility of multiple equilibria?

The left two panels in the next figure describe the effects of a change in m_A, the coefficient of the variable monetary cost. The upper panel shows its effects on $\bar{\gamma}^{Ru}$ and $\bar{\gamma}^{Al}$ whereas the lower one shows the effect on their relative value, $\bar{\gamma}^{Ru}/\bar{\gamma}^{Al}$. The right two panels, instead, concern the change in μ, the fixed monetary cost. It is readily verified from the upper panels that both critical values increase with the two parameters. That is, as the monetary costs rise, the rapid transit city becomes more likely, while the auto city becomes less likely, to be supported as an equilibrium outcome. Furthermore, the relative value, $\bar{\gamma}^{Ru}/\bar{\gamma}^{Al}$, is considered a measure of the likelihood of multiple equilibria. As m_A rises, the value decreases except when m_A is quite low. Therefore, the multiple equilibria become less likely to emerge in a broad range of m_A. As μ rises, $\bar{\gamma}^{Ru}/\bar{\gamma}^{Al}$ decreases when μ is small but increases when it is high. The multiple equilibria become less likely to emerge when μ is low but more likely when it is high.

Figure 5: The effects of changes in monetary costs upon equilibrium patterns

6 Concluding remarks

In this paper, we have discussed the relationship between the spatial structure of a city and the transport mode used there. Paying special attention to their mutual dependence, we have derived two types of equilibria. One is the auto city equilibrium at which workers, distributed over a city thinly, use automobile for their commutes. The other is the rapid transit city equilibrium at which rapid transit services are provided for the commutes of workers, who are distributed densely. We have obtained and characterized the conditions for each type of equilibrium. Furthermore, the possibility of multiple equilibria, that is, the possibility that both types of equilibria emerge for the same set of parameters, has been studied. When such multiple equilibria emerge, the same city may become an auto city or a rapid transit city depending on, for example, historical paths and expectations.

References

[1] Anas, A., R. Arnott, and K. A. Small. (1998) "Urban spatial structure," *Journal of Economic Literature*, 36, 1426-1464.

[2] Baum-Snow, N. (2007) "Did highways cause suburbanization?" *The Quarterly Journal of Economics*, 122, 775-805.

[3] Behrens, K., C. Gaigné, and J.-F. Thisse (2009) "Industry location and welfare when transport costs are endogenous," *Journal of Urban Economics*, 65, 195-208.

[4] Brueckner, J. K. (2001) "Urban sprawl: Lessons from urban economics," in *Brookings-Wharton papers on urban affairs* eds. by Brueckner, J. K., E. Mills, and M. Kremer, Brookings Institution Press, 65-97.

[5] Combes, P.-P., T. Mayer, and J.-F. Thisee. (2008) *Economic geography*, Princeton University Press, Princeton, NJ, USA.

[6] Glaeser, E. L., and M. E. Kahn. (2004) "Sprawl and urban growth," in *Handbook of Regional and Urban Economics* vol. 4 eds. by Henderson, J. H., and J.-F. Thisse, Elsevier, Amsterdam, Netherland, 2481-2527.

[7] Jara-Diaz, S. (2007) *Transport economic theory*, Elsevier, Amsterdam, Netherland.

[8] Fujita, M., P. Krugman, and A. J. Venables. (1999) *The spatial economy: cities, regions, and international trade*, MIT Press, Cambridge, MA, USA.

[9] Roy, R. (2000) (ed.) *Revenues from efficient pricing: Evidence from the member states*, UIC/CER/European Commission, UIC, Paris.

[10] Shams, R. (2005) "Dollar-euro exchange rate 1999-2004: Dollar and euro as international currencies," *HWWA Discussion Paper*, 321.

[11] Small, K. A., and E. T. Verhoef. (2007) *The economics of urban transportation*, Routeledge, New York, NY, USA.

[12] Takahashi, T. (2006) "Economic geography and endogenous determination of transport technology," *Journal of Urban Economics*, 60, 498-518.

[13] Takahashi, T. (2011) "Directional imbalance in transport prices and economic geography," *Journal of Urban Economics*, 69, 92-102.

[14] Train, K., and D. McFadden. (1978) "The goods/leisure trade-off and disaggregate work trip mode choice models," *Transportation Research*, 12, 349-353.

Appendices

Proof of no park-and-ride

Consider a worker who drives a car from her home to $d^0 \leq l$ and then takes the rapid transit from there to the city center. The cost of such an itinerary is equal to $c_A(d - d^0) + c_R(d^0) = (m_R - \lambda)d^0 + \eta_A d + \mu$. If $m_R > \lambda$, it is minimized at $d^0 = 0$ and a worker does not conduct the park-and-ride. If $m_R \leq \lambda$, instead, it is minimized at $d^0 = l$ and she carries out the park-and-ride parking a car at the terminal station.[18]

Now, suppose that $m_R \leq \lambda$. We know that the l-mile inner city workers use rapid transit. Furthermore, the above argument implies that *all the workers living beyond the terminal station* park a car there and then take rapid transit. The total revenue is therefore equal to

$$\Lambda^i_{PKRD}(l, m_R) = m_R \int_0^l \frac{d}{\widetilde{x}^i(d)} \mathrm{d}d + l\, m_R \int_l^{b^i} \frac{1}{\widetilde{x}^i(d)} \mathrm{d}d$$

for $i \in \{A, R\}$. For a given spatial structure, this revenue is increasing in m_R and therefore, maximized at $m_R = \lambda$. When there is no restriction on m_R so that the firm can charge $m_R > \lambda$, on the other hand, it may charge $\widehat{m}(l) > \lambda$ to earn $\Lambda^s(l, \widehat{m}(l))$. If μ is sufficiently high, however,

$$\Lambda^i\big(l, \widehat{m}(l)\big) - \Lambda^i_{PKRD}(l, \lambda) = \frac{\mu}{l} \int_0^l \frac{d}{\widetilde{x}^i(d)} \mathrm{d}d - \lambda l \int_l^{b^s} \frac{1}{\widetilde{x}^i(d)} \mathrm{d}d$$

is positive. In that case, therefore, the firm prefers charging $\widehat{m}(l)$, which discourages the park-and-ride, to charging λ, which gives rise to it. In this paper, we concentrate on this case: μ is sufficiently high that the firm does not charge too low a price that induces the park-and-ride.

Proof of Proposition 1

Let (l^0, m^0) be the solution to $\max\limits_{l \in [0, b^A],\, m_R \geq 0} \Pi^A(l, m_R)$. First, suppose that $\widehat{d}(m^0) < l^0$. Then, $\Pi^A\big(\widehat{d}(m^0), m^0\big) - \Pi^A(l^0, m^0) = \Gamma(l^0) - \Gamma\big(\widehat{d}(m^0)\big) > 0$ since $\Gamma(\cdot)$ is an increasing function. This contradicts the supposition that l^0 maximizes $\Pi^A(l, m_R)$, and consequently $\widehat{d}(m^0) \geq l^0$. Second, suppose that $\widehat{d}(m^0) > l^0$. Then,

$$\Pi^A\big(l^0, \widehat{m}(l^0)\big) - \Pi^A(l^0, m^0) = \Big[\widehat{m}(l^0) - m^0\Big] \int_0^{l^0} \frac{d}{\widetilde{x}^A(d)} \mathrm{d}d > 0,$$

which contradicts the supposition that m^0 maximizes $\Pi^A(l, m_R)$, and consequently, $\widehat{d}(m^0) \leq l^0$. Hence, we have $\widehat{d}(m^0) = l^0$, or $m^0 = \widehat{m}(l^0)$, which implies that $\max\limits_{l \in [0, b^A],\, m_R \geq 0} \Pi^A(l, m_R) = \max\limits_{l \in [0, b^A]} \Omega^A(l) - \Gamma(l)$. The proposition immediately follows from (14).

[18] We arbitrarily assume that workers carry out the park-and-ride when they are indifferent between doing so and not doing so.

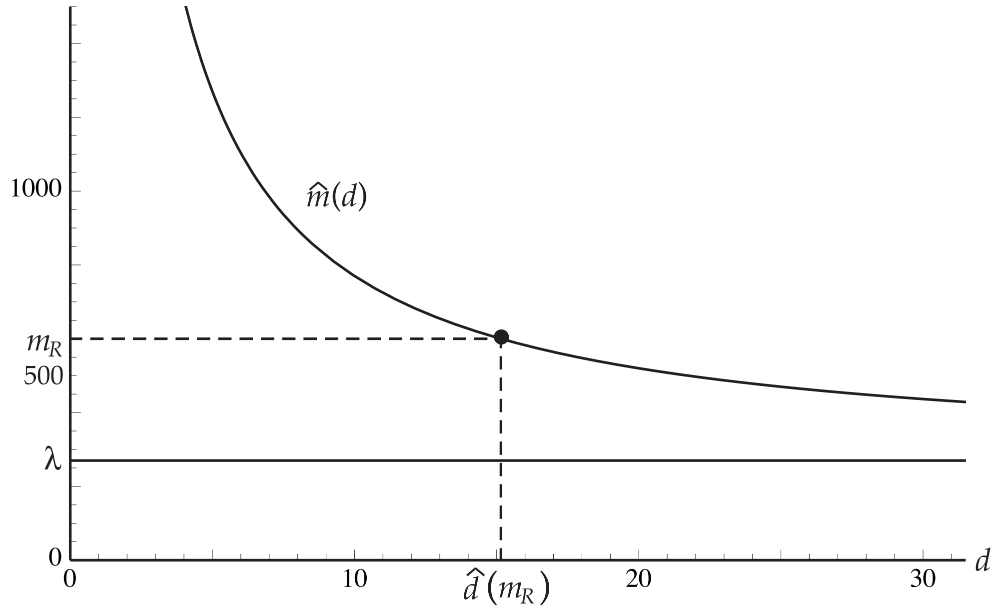


Figure 1: Workers' choices of transport mode

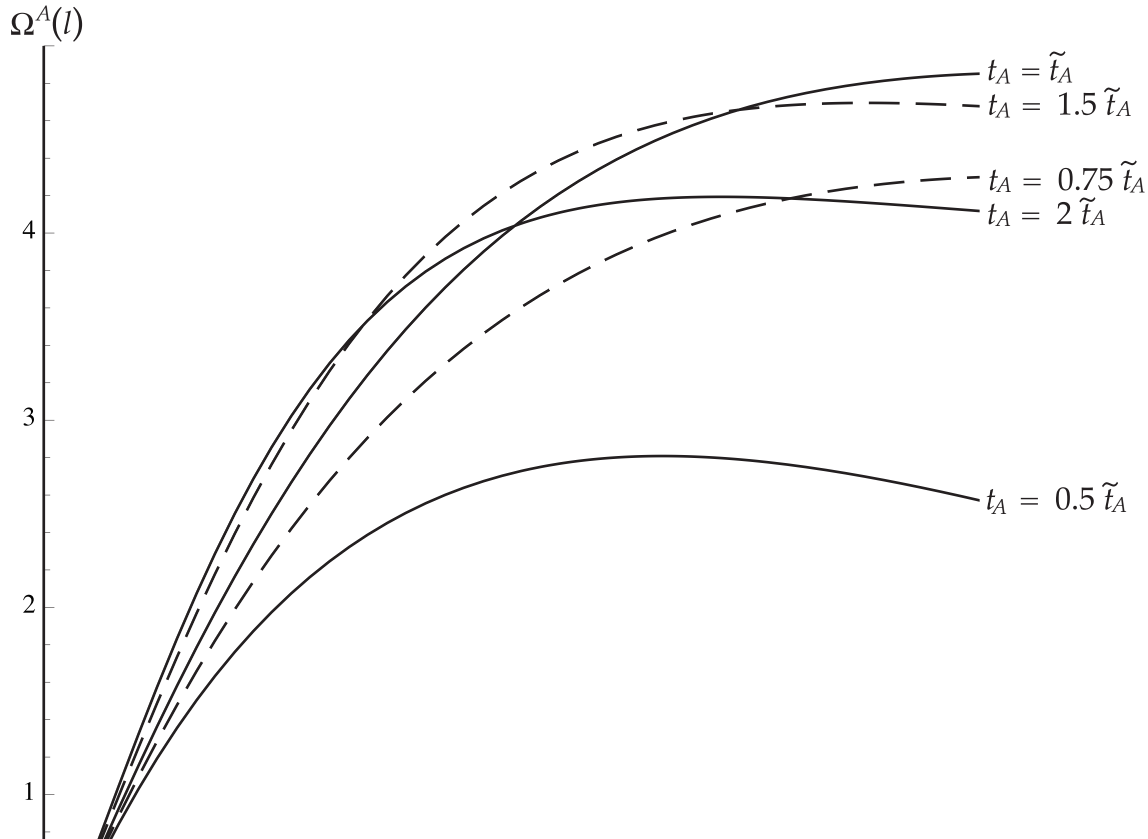


Figure 2: Ω^A revenue curve

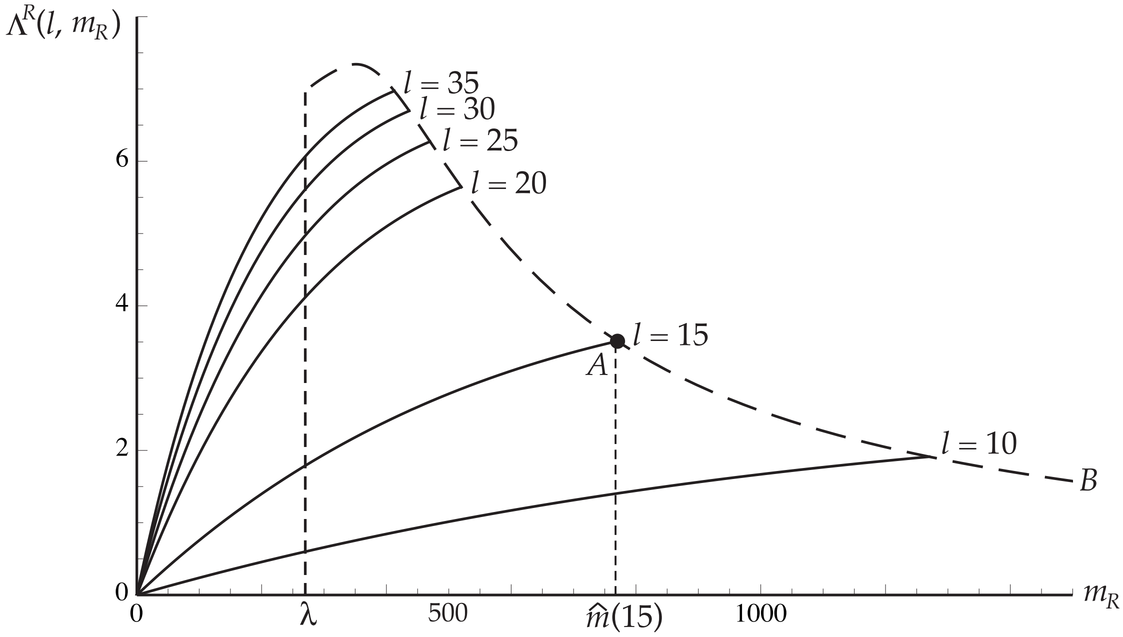


Figure 3: The relationship between the price and the revenue in the rapid transit city

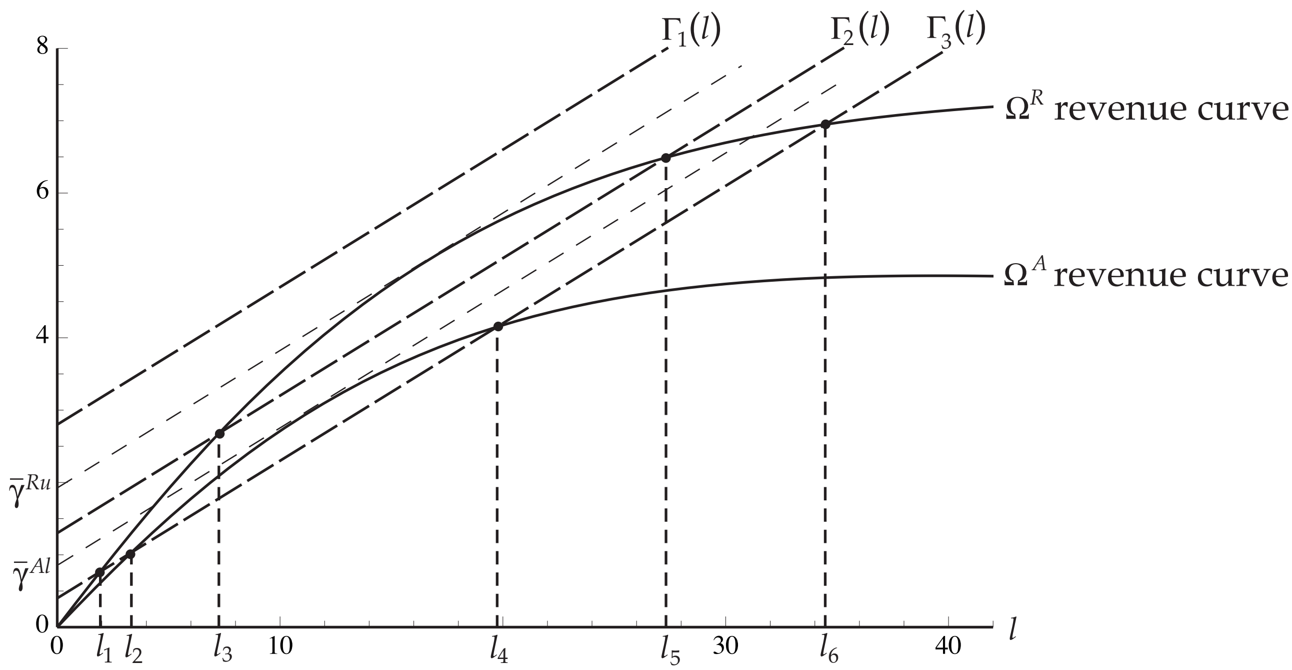
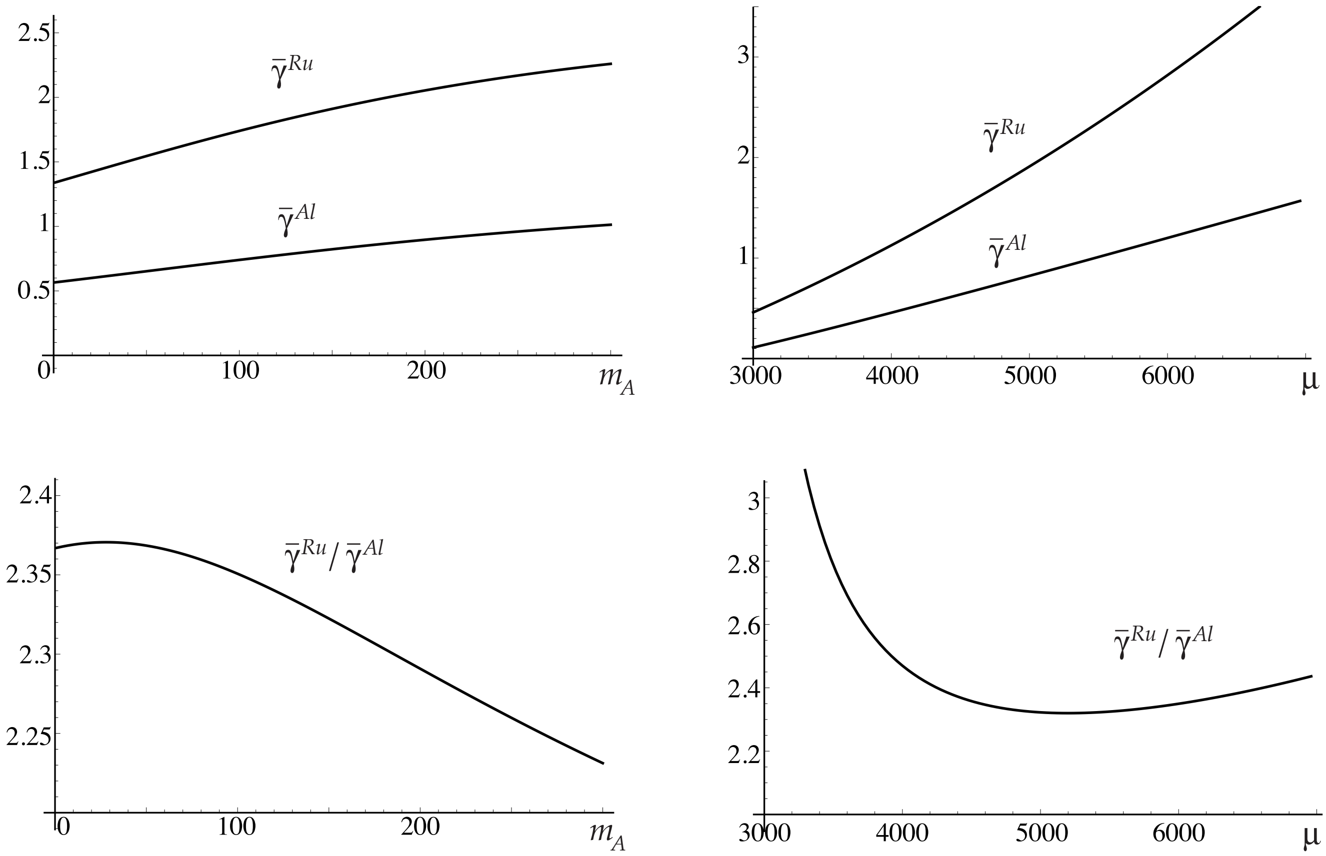


Figure 4: The revenue curves and the cost curve



(a) Change in m_A (b) Change in μ

Figure 5: The effects of changes in monetary costs upon equilibrium patterns