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Endogenous determination of a residential landscape: Asymmetric effects of consumers' choices and multiple equilibria

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Abstract

This paper studies the sizes and rents of dwellings determined as a result of the choices of consumers whose utility depends on the level of landscape amenities in their neighborhood. The level is, in turn, determined by the sizes of dwellings there. It is shown that there emerges a possibility of multiple equilibria if the effect of each consumer's choice upon landscape amenities is asymmetric.

Keywords: landscape; multiple equilibria; rent gradient; spatial structure of a city; urban amenities; urban mosaic

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1 Introduction

One of the most significant features of a city is a spatial diversity within it. Literature in urban economics has traditionally been emphasizing a geographical variation based on the distance from a city center. However, diversity is observed not only for the spaces at different distances but also for those at quite similar distances. We often encounter, for instance, the situation in which neighboring districts in a city, only divided by a narrow street, exhibit contrasting characteristics with respect to land use patterns. To give another example, it is not rare that the western part and the eastern part, say, of the same city are thoroughly different from each other in many aspects. For an illustrative purpose, let us look at the data for the Nagoya metropolitan area in Japan, which is the third largest metropolitan area in that country. Table 1 shows means, standard deviations and coefficients of variation for the rents par 1 square meter of floor space for privately-owned rental apartments and detached houses, classified by the distances from the city center.[1] It reveals that a considerable variation exists for dwellings within the same distance classes. Of course, this result is presented only to motivate readers; and if one wanted to demonstrate the spatial diversity among the spaces at similar distances from a city center more rigorously, it would be necessary to conduct a more precise analysis such as the one based on a hedonic approach.

Table 1: Rents of privately-owned rental apartments and detached houses in the Nagoya metropolitan area

distance from a city center (km)	number of samples	mean (yen/m^2)	standard deviation (yen/m^2)	coefficient of variation
0-10	317,800	1671.73	775.41	0.464
10-20	170,200	1366.35	585.29	0.428
20-30	127,500	1263.51	546.26	0.432
30-40	151,500	1252.71	543,70	0.434
40-50	31,300	1216.85	543.41	0.447

Source: Made from *Housing and Land Survey* by Statistics Bureau (2003).

Most important reasons for such a diversity are arguably differences in income levels and preferences among the residents of each district (Glaeser et al. (2008) and Bruekner and Rosenthal (2009)). Some neighborhoods, for instance, turn out to attract the rich who can afford to pay higher rents while some others end up accommodating the poor who pay lower rents, even if they are located at similar distances from a city center. In other cases, families with kids prefer spacious dwellings while singles and the elderly opt for more compact ones. Notwithstanding, the phenomenon of the "urban mosaic" might have much deeper roots, because casual observations

[1] In the original data of Statistics Bureau (2003), rents are divided into 18 classes. We compute statistics using the medians of respective classes.

suggest that it can emerge even when consumers in the two neighborhoods at similar distances from a city center are more or less homogenous in terms of income and preference. If this is true, it would be better to understand the phenomenon as a realization of multiple equilibria. This paper is an attempt to do so.

For that purpose, we pay attention to the role of urban amenities. A number of studies have argued that they considerably affect consumers' locational choices in a city (see Rosen (1974), Rosen (1979), Roback (1982), Bartik and Smith (1987), Papageorgiou and Pines (1999) and Cho (2001) among others). For some of them such as natural or environmental amenities like warm climate, and historical amenities like old religious buildings, palaces and castles, their levels are given. For others, in contrast, they are determined as a consequence of consumers' choices.[2] In particular, there are a type of amenities whose levels directly depend on the sizes of land lots and dwellings. A typical example is a residential landscape: a sequence of large detached houses occupying wide land lots with a mass of greenery undoubtedly constitutes an amenity good that will raise the utility of nearby residents.

This type of amenity good gives rise to a possibility of multiple equilibria. At one equilibrium, residents live in large detached houses constructed within large land lots whereas at another equilibrium, they live in small dwellings with small land lots. A higher level of landscape amenities is generated at the sparsely populated equilibrium than the densely populated one, which favors consumers at the former equilibrium than the latter, other things being equal. If consumers are homogeneous, they receive an equal level of utility at an equilibrium. Therefore, the rents of dwellings and land lots must be higher at the sparsely populated equilibrium, other things being equal. In other words, the rents at the two equilibria are determined so that the positive effect of a higher level of landscape amenities at the sparsely populated equilibrium exactly offsets the negative effect of a higher rent at that equilibrium.

To explore this possibility of multiple equilibria, we construct a simple Alonso-Mills-Muth type model extended to incorporate landscape amenities. Two elements are important. First, the level of landscape amenities is determined by the choices of the consumers in a neighborhood. Second, each consumer makes a decision taking into account the effect of her own choice upon landscape amenities. Main finding is that whether multiple Nash equilibria exist or not depends on the nature of landscape amenities. To understand this point, suppose that all the residents in a neighborhood consume an equal amount of housing and then one consumer changes the amount of housing consumption by one unit. If the landscape amenities are symmetric, that is, if the rise of amenity level due to the one-unit increase in her consumption is equal to the decline of amenity level due to the one-unit decrease in her consumption, then, there exist no multiple

[2]One of the most important examples would be local public goods such as highways, mass transit, museums and good schools. Many researchers have been studying how their amounts of provision are determined and how it affects consumers' location choices in a city. See, for example, Wright (1977), Polinsky and Rubinfeld (1978), Helpman and Pines (1980) and Sullivan (1985).

equilibria, provided that some regularity conditions on consumers' preference are satisfied. If they are asymmetric and the rise is smaller than the decline, in contrast, there do exist multiple equilibria. In that case, a certain range of the amounts of housing consumption and that of the price of housing are supported by an equilibrium.

One implication of the multiple equilibria is that a land rent gradient may become positive, that is, land rent may *rise* as one moves away from a city center. This possibility has been an interest of some researchers. Richardson (1977), for instance, attempts to demonstrate it in a setting where a sparse distribution of consumers generates positive externalities. Re-examining this problem, however, Grieson and Murray (1981) and Tauchen (1981) show that there is no such possibility as long as consumers' preference is well-behaved. Their result corresponds to our finding that no multiple equilibria exist when landscape amenities are symmetric.[3]

In addition to these works, there are several works that discuss the endogenous formation of urban amenities. First, Schall (1976) and Henderson (1985) study the impact of the quality of neighborhood upon consumers' choices. In their studies, the quality is defined as an average level of housing services consumed by neighbors. Second, Brueckner et al. (1999) examine the role of urban amenities on the location of different income groups within a city. They consider not only the amenity goods whose amounts are exogenous but also the amenity goods whose amounts depend on local income levels like restaurants, theaters and sports facilities. Whether the rich are located in a central city or a suburb depends on the relative importance of the two types of amenity goods. Third and finally, Walsh (2007) examines the amounts of open spaces in cities through an empirical general equilibrium approach, in which open spaces "arise 'naturally' as a result of land market outcomes."[4]

2 Basic framework

We consider a monocentric city in a homogeneous plane, which is made up of a number of small neighborhoods. Consumers in the city work at a city center to earn a given wage rate, w. The commuting costs for a consumer living at distance d from the city center are equal to $t(d)$ with $t'(d) > 0$ where $t'(\cdot)$ denotes the derivative of $t(\cdot)$ (we will use similar notations for the derivatives of other variables). Consumers' income consists of only wage: they do not own dwellings nor lands and all the rents are received by absentee landlords. Thus, the net income of each consumer is $y(d) \equiv w - t(d)$.

Consumers consume housing and a composite good. Here, "housing" contains both a dwelling and the land lot upon which it is built. Although they are actuallly characterized by many el-

[3]More specifically, they concentrate on the case where the level of landscape amenities depends on an *average* lot size. In this case, amenities are symmetric.

[4]Yang and Fujita (1983) study the impact of open spaces on the spacial structure of a city; but their distribution is assumed to be exogenous.

ements such as size, the number of floors, height, building materials, colors and design for a dwelling, and size, shape, location and the width of the road it faces for a land lot, we summarize them by one variable x. Without loss of generality, it is defined so that larger x is associated with better characteristics. As a natural interpretation, we regard x as a size of a dwelling.

Consumers' utility depends not only on the size of a dwelling and the amount of a composite good but also on the quality of living environment or the amenities in their neighborhood. Thus, we denote the utility of consumer i as $u(x_i, z_i, a)$, where x_i and z_i denote the size of a dwelling and the amount of a composite good consumed by that consumer, respectively, and a denotes the level of amenities in her neighborhood. The utility function is assumed to be increasing in each argument (a is defined so that a higher value corresponds to a better environment).

We focus on the amenities whose level depends on the sizes of dwellings in a neighborhood. A typical example is landscape amenities: a sequence of larger dwellings contributes to a creation of better scenery and raises the utility of residents. Then, the level of amenities is written as $a(x_1, x_2, ..., x_n)$, where n is the number of consumers in a neighborhood. Function $a(\cdot)$ is continuous and nondecreasing in each variable (the landscape is no worse when one dwelling is bigger and the others remain of the same size).

A twist of our model is that the amenity function can be non-differentiable with respect to x_i at $x_i = x_j$ for $j \neq i$, while it is differentiable at any other point. It is often observed, for instance, that a house slightly shabbier than neighbors' gives a devastating impact on the environment of a neighborhood although a house slightly more handsome than them does not contribute much to amenities. Here, consumers value a loss more significantly than the equivalent gain, which is one of the most important observations in the behavioral economics.[5] In this case of *loss aversion*, the left derivative of $a(\cdot)$ exceeds the right derivative. The opposite case where the left derivative is smaller than the right derivative is theoretically conceivable; but we do not pay much attention to that case for the following two reasons. First, it is difficult to find the evidences for the consumers' behaviors to care about a benefit more than a loss. Second, it turns out that there exists no equilibrium in that case, as we will see later. Therefore, we concentrate on the case where

$$\lim_{x_i \to x_j^-} a_i(x_1, ..., x_n) \geq \lim_{x_i \to x_j^+} a_i(x_1, ..., x_n). \tag{1}$$

Here, a_i denotes the partial derivative of $a(\cdot)$ with respect to the i-th argument (similar notations will be used for the partial derivatives of other functions).

We limit our analysis to the symmetric equilibria at which consumers choose dwellings of the same size. Thus, consider, without loss of generality, the decision problem of consumer 1 when all her neighbors consume the dwellings of sizes x^0. Then, the amenity function can be written as $a(x, \underbrace{x^0, \cdots, x^0}_{n-1}) \equiv \alpha(x, x^0)$. Since $a(\cdot)$ is nondecreasing in each variable, the same remark applies to $\alpha(\cdot)$. Moreover, the assumption on the differentiability of $a(\cdot)$ mentioned above implies that

[5]This point is mainly discussed by the prospect theory originating from Kahneman and Tversky (1979) and Tversky and Kahneman (1992). For the overview of the topic, see Starmer (2000).

$\alpha(x,x^0)$ is differentiable with respect to x at any point with $x \neq x^0$, but may not be so at $x = x^0$. If the amenity function satisfies (1), the following equation holds.

$$\lim_{x \to x^{0-}} \alpha_1(x,x^0) \geq \lim_{x \to x^{0+}} \alpha_1(x,x^0). \tag{2}$$

We impose several additional assumptions on the amenity function. First, when all the consumers choose dwellings of the same size, the level of amenities is given by $\alpha(x,x) \equiv A(x)$. Our assumption is that $A(\cdot)$ is differentiable, that is,

$$\lim_{x \to x^{0-}} \alpha_1(x,x^0) + \lim_{x \to x^{0+}} \alpha_2(x^0,x) = \lim_{x \to x^{0+}} \alpha_1(x,x^0) + \lim_{x \to x^{0-}} \alpha_2(x^0,x) = A'(x^0) \tag{3}$$

for any x^0. Because $\alpha(\cdot)$ is nondecreasing in each argument, $A(\cdot)$ is a nondecreasing function. Second, we assume that $\lim_{x \to x^{0-}} \alpha_1(x,x^0)$ and $\lim_{x \to x^{0+}} \alpha_1(x,x^0)$ are differentiable with respect to x^0. Third and last, $\alpha(x,x^0)$ is concave in x, that is, $\alpha_{11}(x,x^0) \leq 0$, where $\alpha_{11}(x,x^0) \leq 0$ is a partial derivative of $\alpha_1(x,x^0)$ with respect to the first argument (similar notations will be used for the other second-order partial derivatives).

Before completing the description of a basic setting, to take a look at two examples of the amenity function would help us understand the framework.

Example 1

As a benchmark, consider the amenity function given by $a(x_1,...,x_n) = f\left(\sum_{i=1}^{n} x_i \Big/ n\right)$, where $f(\cdot)$ is differentiable, increasing and concave. In this example, the level of amenities is determined by the average size of dwellings in a neighborhood. We refer to this type of amenity function as an *average-value amenity function*. This function is differentiable with respect to each x_i: $\lim_{x_i \to x_j^-} a_i(x_1,...,x_n) = \lim_{x_i \to x_j^+} a_i(x_1,...,x_n) = \dfrac{1}{n} f'\left(\sum_{i=1}^{n} x_i \Big/ n\right)$ for $i = 1,...,n$ and $j = 1,...,n$ with $j \neq i$. Furthermore, $\alpha(x,x^0) = f\left(x^{AVE}\right)$ where $x^{AVE} \equiv \left[x + (n-1)x^0\right]/n$. Thus, $\lim_{x \to x^{0-}} \alpha_1(x,x^0) = \lim_{x \to x^{0+}} \alpha_1(x,x^0) = f'(x^0)/n$. In addition, $A(x) = f(x)$, which is differentiable, increasing and concave for any x by assumption. Finally, $\alpha_{11}(x,x^0) = f''(x^{AVE})/n^2 \leq 0$.

Example 2

Next, we turn to another example which illustrates well the unique feature of our model. The amenity function is given by

$$a(x_1,...,x_n) = g\big(\min[x_1,...,x_n]\big), \tag{4}$$

where $g(\cdot)$ is differentiable, increasing and concave. In this example, the level of amenities is determined by the size of the smallest dwelling in a neighborhood. It depicts an extreme case where one blot spoils everything. We refer to this type of amenity function as a *minimum-value amenity function*. Three observations follow. First, let $m_{-i} \equiv \min\left[x_1,...,x_{i-1},x_{i+1},...,x_n\right]$. Then, $a_i(x_1,...,x_n)$ is equal to $g'(x_i) > 0$ if $x_i < m_{-i}$, and to 0 if $x_i > m_{-i}$. Consequently,

$$\lim_{x_i \to m_{-i}^-} a_i(x_1,...,x_n) = g'(x_i) > 0 = \lim_{x_i \to m_{-i}^+} a_i(x_1,...,x_n)$$

and therefore, (1) is satisfied. Second, $\alpha(x, x^0) = g\big(\min[x, x^0]\big)$, which implies that $\alpha_1(x, x^0)$ is equal to $g'(x)$ if $x < x^0$, and to 0 if $x > x^0$. Consequently,

$$\lim_{x \to x^{0-}} \alpha_1(x, x^0) = g'(x^0) > 0 = \lim_{x \to x^{0+}} \alpha_1(x, x^0).$$

Third, $A(x) = g(x)$, which is differentiable, increasing and concave for any x. Last, $\alpha_{11}(x, x^0) \leq 0$ since it is equal to $g''(x)$ if $x < x^0$, and to 0 if $x > x^0$.

Returning to the basic framework, let r be the rent of a dwelling and take the composite good as a numéraire. The budget constraint of the i-th consumer in a neighborhood at distance d from the city center is given by

$$rx_i + z_i = y(d).$$

Finally, consumers can freely migrate among neighborhoods, and the city is small and open. Therefore, the level of utility they enjoy is a constant and given. We denote it by $\bar{u}$.

3 Equilibrium in a general setting

Because the analysis is limited to symmetric equilibria, it is useful to denote the utility function by $v(x, x^0, r) \equiv u\Big(x,\ y(d) - rx,\ \alpha(x, x^0)\Big)$. The Nash equilibrium is defined as a pair (x^*, r^*) that satisfies the following two conditions:

$$v(x^*, x^*, r^*) \geq v(x, x^*, r^*) \tag{5}$$

for any $x \geq 0$ and

$$v(x^*, x^*, r^*) = \bar{u}. \tag{6}$$

The first condition requires that each consumer maximize her utility given the neighbors' choices. The second prescribes that the resulting utility level equal the one prevailing in the economy. Note that each consumer takes into account not only the effects of neighbors' choices but also the effects of her own choice upon the amenity level. This is a significant departure from the existing works mentioned in the introduction, which implicitly assume that consumers disregard their own effects.

First of all, since $\alpha(x, x^0)$ is differentiable with respect to x at $x \neq x^0$,

$$v_1(x, x^0, r) = u_x - ru_z + u_a\alpha_1(x, x^0) \quad \text{for } \ x \neq x^0.$$

Here, u_x, u_z and u_a denote the partial derivatives of $u(x, z, a)$ with respect to x, z and a, respectively, evaluated at $(x, z, a) = \big(x,\ y(d) - rx,\ \alpha(x, x^0)\big)$. To avoid unnecessary complications, we assume that function $v(x, x^0, r)$ is strictly concave in x as long as $x \neq x^0$. In other words, the following assumption is imposed for $v_{11}(x, x^0, r)$.

Assumption 1

$v_{11}(x, x^0, r) < 0$ *for any* $x \neq x^0$.

This assumption says that the marginal utility of housing consumption, *inclusive of the effects of the induced change in the amenity level*, decreases as the dwelling becomes bigger.

For $x = x^0$, on the other hand, it is necessary to distinguish the left and right derivatives:

$$\begin{cases} \lim\limits_{x \to x^{0-}} v_1(x, x^0, r) = u_x - r u_z + u_a \lim\limits_{x \to x^{0-}} \alpha_1(x, x^0) \equiv \lambda(x^0) \\ \lim\limits_{x \to x^{0+}} v_1(x, x^0, r) = u_x - r u_z + u_a \lim\limits_{x \to x^{0+}} \alpha_1(x, x^0) \equiv \mu(x^0), \end{cases} \tag{7}$$

where u_x, u_z and u_a are evaluated at $(x, z, a) = \left(x^0, y(d) - r x^0, \alpha(x^0, x^0)\right)$. (Although $\lambda(\cdot)$ and $\mu(\cdot)$ depend on r, we suppress r to make the presentation clearer.)

We are now ready to discuss the utility maximization problem of (5). Because $v(x, x^0, r)$ is differentiable with respect to x and strictly concave in x for any $x \neq x^0$, the necessary and sufficient condition for the utility maximization is that $v_1(x, x^*, r) > 0$ for any $x < x^*$ and $v_1(x, x^*, r) < 0$ for any $x > x^*$. By construction, these conditions hold if and only if

$$\lambda(x^*) \geq 0 \quad \text{and} \quad \mu(x^*) \leq 0. \tag{8}$$

We have mentioned earlier that no equilibrium exists unless (1) is satisfied. This can be readily verified: $\lambda(x^0) < \mu(x^0)$ for any x^0 and no x^* satisfies (8), if (2) does not hold.

To characterize the solutions, we derive the derivative of $\lambda(\cdot)$:

$$\begin{aligned} \lambda'(x^0) &= \lim_{x \to x^{0-}} v_{11}(x, x^0) + \lim_{x \to x^{0-}} v_{12}(x, x^0) \\ &= \lim_{x \to x^{0-}} v_{11}(x, x^0) + \lim_{x \to x^{0-}} \alpha_2(x, x^0) \cdot \lim_{x \to x^{0-}} \omega^a(x, x^0) + u_a \lim_{x \to x^{0-}} \alpha_{12}(x, x^0), \end{aligned}$$

where $\omega^k(x, x^0)$ is the change in the marginal utility of housing consumption caused by a rise in $k \in \{x, z, a\}$: $\omega^k(x, x^0) \equiv u_{xk} - r u_{zk} + u_{ak} \alpha_1(x, x^0)$. To understand this equation, let us focus on a particular consumer thinking of buying a dwelling whose size is almost equal to the neighbors'. First, when neighbors' dwellings are bigger, the dwelling she has a mind to buy is also bigger, which affects her marginal utility. This *own effect*, which is negative due to Assumption 1, is represented by the first term in the second line. Note that it includes the effect of the amenity improvement stemming from the increase in her own housing consumption, namely, an own amenity improvement effect.[6] Second, the increase in the sizes of neighbors' dwellings directly raises the amenity level in the neighborhood. This effect is captured by the second term in the second line. Third and last, the sizes of neighbors' dwellings affect the magnitude of the own amenity improvement effect mentioned above. This is represented by the last term in the second line. Because the signs of the last two terms describing *cross effects* are ambiguous, we cannot determine the direction of the total effect. In this paper, however, we confine our attention to the case where the own effect is strong enough to dominate the cross effects when $\lambda(x^0) = 0$. In other words, we assume that $\lambda(x)$ is decreasing at the value of x that solves $\lambda(x) = 0$. We assume a similar property for $\mu(\cdot)$.

[6] Here, $\lim\limits_{x \to x^{0-}} v_{11}(x, x^0, r) = \lim\limits_{x \to x^{0-}} \left[\omega^x(x, x^0) - r\omega^z(x, x^0) + \omega^a(x, x^0)\alpha_1(x, x^0) + u_a \alpha_{11}(x, x^0)\right]$. The last term in the square brackets represents the own amenity improvement effect.

Assumption 2

$$\lambda'(x) < 0 \quad \textit{if } x \textit{ solves } \lambda(x) = 0; \quad \textit{and} \quad \mu'(x) < 0 \quad \textit{if } x \textit{ solves } \mu(x) = 0.$$

The amenity function discussed in Example 2 satisfies this assumption as long as Assumption 1 holds. We will discuss this point later. Figure 1 describes $\lambda(x)$ and $\mu(x)$ as a left derivative curve and a right derivative curve, respectively. Because $\lambda(x) \geq \mu(x)$ for any x, the former curve lies above or on the latter.

Figure 1: Left and right derivative curves

Assumption 2 implies that $\lambda(x) = 0$ ($\mu(x) = 0$, resp.) has at most one solution, which we denote as a function of r by $\bar{x}(r)$ ($\underline{x}(r)$, resp.). Two comments are worth adding. First, $\lambda(x) \geq \mu(x)$ implies that $\underline{x}(r) \leq \bar{x}(r)$. Second,

$$\bar{x}'(r) = \frac{u_z}{\lambda'(x)} < 0 \qquad \text{and} \qquad \underline{x}'(r) = \frac{u_z}{\mu'(x)} < 0.$$

Furthermore, (8) is equivalent to

$$x^* \in \left[\underline{x}(r^*),\ \bar{x}(r^*)\right]. \tag{9}$$

The interval in (9) is the set of the sizes of dwellings that are compatible with consumers' optimizing behaviors given their rents, and thus referred to as a *compatibility set*. Figure 2 shows the lower and upper limits of this set in an x-r space as lower and upper *compatibility curves*, respectively. Both curves have negative slopes.

Figure 2: Compatibility curves and sustainability curve

Example 2-2

Here, it would be instructive to revisit the example with the minimum-value amenity function, (4). Because $\lim_{x \to x^{0-}} \alpha_2(x, x^0) = \lim_{x \to x^{0-}} \alpha_{12}(x, x^0) = 0$ for this function, we have $\lambda'(x^0) = \lim_{x \to x^{0-}} v_{11}(x, x^0, r)$. Therefore, as long as Assumption 1 is satisfied, Assumption 2 holds. Furthermore, (8) becomes $-u_a g'(x^*) \leq u_x - r^* u_z \leq 0$, where all the partial derivatives are evaluated at $x = x^*$. $\bar{x}(r^*)$ is a solution to $u_x - r^* u_z = -u_a g'(x)$ while $\underline{x}(r^*)$ is a solution to $u_x - r^* u_z = 0$.

Next, let us turn to the second condition for an equilibrium, (6). It implicitly yields x^* as a function of r^*:

$$x^* = s(r^*). \tag{10}$$

Here, function $s(\cdot)$ gives the size of a dwelling that brings a consumer the utility level of $\bar{u}$ for a given rent. If (x, r) does not satisfy $x = s(r)$, it invokes consumers' migration from or into the city. In this sense, (10) assures the *sustainability* of a city's population.

To see that the sustainability function $s(\cdot)$ is increasing, let us introduce an additional notation, $V(x, r) \equiv v(x, x, r)$. It represents the utility level of the consumer who chooses a dwelling of the same size as her neighbors'. Then, we have

$$s'(r^*) = \frac{x^* u_z}{V_1(x^*, r)} = \frac{x^* u_z}{u_x - r^* u_z + u_a A'(x^*)} > 0. \tag{11}$$

The denominator, which measures the effects of an increase in x^* upon the utility level, is decomposed into two parts. First, $u_a A'(x^*)$ represents an indirect effect through the improvement of amenities. This *amenity effect* is positive. Second, $u_x - r^* u_z$ represents a *direct effect*. The second inequality in (8) implies that $u_x - r^* u_z \leq 0$ at $(x, z, a) = \big(x^*, y(d) - r^* x^*, \alpha(x^*, x^*)\big)$ because $\lim_{x \to x^{0+}} \alpha_1(x, x^0) \geq 0$ (see (7)): the direct effect is nonpositive. The sign of the denominator is determined by the relative sizes of the two effects. However, (3) implies that $A'(x^*) \geq \lim_{x \to x^{*-}} \alpha_1(x, x^*)$. Therefore, it follows from the first inequality of (8) that the denominator is nonnegative, that is, $V_1(x^*, r^*) \geq 0$: the amenity effect offsets or more than offsets the direct effect. Hence, the sustainability function is increasing and, in Figure 2, described by an upward sloping curve, which is referred to as a *sustainability curve*.

The set of the equilibrium pairs, namely, the (x^*, r^*)'s that satisfy both (9) and (10), is represented by a part or parts of the sustainability curve between the upper and lower compatibility curves. Because both the compatibility curves are downward sloping and the sustainability curve is upward sloping, the latter cuts each of the former at most once. We denote these intersections by (x^l, r^l) and (x^u, r^u), respectively (see Figure 2), that is, (x^l, r^l) solves $x^l = \underline{x}(r^l)$ and $x^l = s(r^l)$, whereas (x^u, r^u) solves $x^u = \bar{x}(r^u)$ and $x^u = s(r^u)$. Then, the set of equilibrium pairs is given by

$$E \equiv \left\{ (x, r) \middle| x = s(r),\ x \in [x^l,\ x^u] \right\}.$$

Thus, x^l and r^l give the *lower bounds* of the equilibrium values of x and r, respectively, whereas x^u and r^u give their *upper bounds*.

Set E does not degenerate to a point as long as the lower compatibility curve always lies below the upper compatibility curve. Instead, if the two curves coincide with each other, the set is reduced to a point. Thus, we have established the following result.

Proposition 1 (Multiple equilibria)
i) Suppose that $\alpha(x, x^0)$ is not *differentiable with respect to x, that is, (2) holds with the inequality, at $x = x^0$ for any x^0. If there is an equilibrium, there always exists another equilibrium.*
ii) Instead, suppose that $\alpha(x, x^0)$ is differentiable with respect to x, that is, (2) holds with the equality, at $x = x^0$ for any x^0. Then, the equilibrium, if it exists, is unique.

The second part of this proposition says that without the non-differentiability of the amenity function, merely introducing externalities in amenity formation does not produce multiple equilibria, provided that the utility function and the amenity function satisfy the regularity conditions (Assumptions 1 and 2). In short, the multiple equilibria necessitate the non-differentiability.

Suppose that there are multiple equilibria. Then, two equilibrium pairs of (x, r) must be on the same sustainability curve, which is upward sloping. Therefore, r is higher at the equilibrium with larger x than at the equilibrium with smaller x.

Proposition 2 (Relationship between the sizes and rents of a dwelling)
Suppose that (x', r') and (x'', r'') are equilibrium pairs for given d. Then, $r' \left\{ \begin{array}{c} \geqq \\ < \end{array} \right\} r''$ if $x' \left\{ \begin{array}{c} \geqq \\ < \end{array} \right\} x''$.

In a standard setting with no amenities, in order to keep the utility at the same level, it is necessary to reduce housing consumption when its price rises. However, this is not always the case in our model. Despite the rise in the rent, consumers may increase housing consumption because doing so improves amenities: the effect of the decrease in the consumption of the other goods may be more than compensated by the effect of the amenity improvement.

It is worth adding a comment on the level of social welfare. As far as only the consumers living in the city are concerned, the multiple equilibria give the same level of welfare, because they enjoy the same level of utility. However, if we take into account absentee landlords, the level of social welfare will depend on the rents of dwellings: the equilibrium with a higher rent (and a larger dwelling) yields a higher welfare level.

How does a space affect the range of the equilibria? To answer this question, note that in our model, the distance from the city center affects the equilibrium range only through the level of disposable income, $y(d)$. Therefore, to discuss the effects of a change in d, it suffices to examine the effects of a change in $y(d)$.

To begin with, note that since $y(\cdot)$ is a continuous function, all x^l, r^l, x^u and r^u are also continuous with respect to d. Consquently, if multiple equilibria exist at a location, then, they also exist at sufficiently close locations. Therefore, *a rent gradient can be positive in a sufficiently narrow range of geographical space*. This would be one of alternative explanations for the positive rent gradient discussed by many researchers (Schall (1976), Richardson (1977), Grieson and Murray (1981), Tauchen (1981) and Henderson (1985)), although the problem of indeterminacy arises.

Furthermore, totally differentiating $x^l = \bar{x}(r^l)$ and $x^l = s(r^l)$ with respect to x^l, r^l and $y(d)$, and solving the two resulting equations, we derive $\mathrm{d}x^l/\mathrm{d}y(d)$ and $\mathrm{d}r^l/\mathrm{d}y(d)$. Similarly, totally differentiating $x^u = \bar{x}(r^u)$ and $x^u = s(r^u)$ yields $\mathrm{d}x^u/\mathrm{d}y(d)$ and $\mathrm{d}r^u/\mathrm{d}y(d)$:

$$\frac{\mathrm{d}x^k}{\mathrm{d}d} = -t'(d)\frac{\mathrm{d}x^k}{\mathrm{d}y(d)} = -\frac{t'(d)u_z s'(r^k)}{x^k\Gamma^k} \qquad \text{and} \qquad \frac{\mathrm{d}r^k}{\mathrm{d}d} = -\frac{t'(d)}{x^k\Gamma^k}(\Gamma^k + u_z)$$

for $k \in \{l, u\}$, where

$$\begin{cases} \Gamma^l \equiv \mu'(x^l)s'(r^l) - x^l \lim_{x \to x^{l+}} \omega^z(x, x^l, r) - u_z, & \text{and} \\ \Gamma^u \equiv \lambda'(x^u)s'(r^u) - x^u \lim_{x \to x^{u-}} \omega^z(x, x^u, r) - u_z. \end{cases}$$

First, let us examine the changes in x^l and x^u. Becuase $\mu'(x^l) < 0$ by Assumption 2 and $s'(r^l) > 0$, Γ^l is negative if $\lim_{x \to x^{l+}} \omega^z(x, x^l, r)$ is nonnegative. Even if $\lim_{x \to x^{l+}} \omega^z(x, x^l, r)$ is negative, however, Γ^l is still negative as long as it is not too large in the absolute value. In such cases, we have $\mathrm{d}x^l/\mathrm{d}d > 0$: the lower bound of the equilibrium sizes of dwellings rises as we move farther away from the city center. The similar remark applies to the upper limit.

Next, we examine the changes in r^l and r^u. When $\lim_{x \to x^{l+}} \omega^z(x, x^l, r)$ is nonnegative or its absolute value is sufficiently small, $\Gamma^l + u_z$ as well as Γ^l becomes negative. In that case, $\mathrm{d}r^l/\mathrm{d}d < 0$: the lower bound of the equilibrium rents declines as we go farther away from the city center. Once again, the similar remark applies to the upper limit. Here, note that $\omega^z(x, x^k, r)$ represents the change in the marginal utility of housing consumption when the income level y changes. Our findings are summarized as follows:

Proposition 3 (Spatial variation)
Suppose that the marginal utility of housing consumption increases or only slightly decreases as the income level rises. Then, as we move farther away from the city center, both the lower and upper bounds of the equilibrium sizes of dwellings rise and both the lower and upper bounds of the equilibrium rents decline.

We can explain these results using Figure 2 above. First of all, look at the sustainability curve. By totally differentiating (6) with respect to x^* and y while fixing r^*, we obtain

$$\frac{\mathrm{d}x^*}{\mathrm{d}d} = \frac{t'(d)s'(r^*)}{x^*} > 0,$$

where (11) is used. Therefore, the sustainability curve shifts to the right as d increases. Next, we examine the compatibility curves. Remember that $\underline{x}$ is defined as a solution to $\mu(\underline{x}) = 0$ (in the rest of this section, the arguments r's of $\underline{x}$ and of $\bar{x}$ are suppressed for the sake of clarity). Totally differentiating this equation with respect to d yields

$$\mu'(\underline{x})\frac{\mathrm{d}\underline{x}}{\mathrm{d}d} - t'(d) \lim_{x \to \underline{x}^+} \omega^z(x, \underline{x}, r) = 0.$$

Since $\mu'(\underline{x}) < 0$ by Assumption 2, $\mathrm{d}\underline{x}/\mathrm{d}d \leq 0$ as long as $\lim_{x \to \underline{x}^+} \omega^z(x, \underline{x}, r) \geq 0$: the lower compatibility curve shifts to the left as d rises. Similarly, because

$$\lambda'(\bar{x})\frac{\mathrm{d}\bar{x}}{\mathrm{d}d} - t'(d) \lim_{x \to \bar{x}^-} \omega^z(x, \bar{x}, r) = 0,$$

the upper compatibility curve also shifts to the left as long as $\lim_{x \to \bar{x}^-} \omega^z(x, \bar{x}, r) \geq 0$. The curves associated with higher d are shown by the two downward-sloping dotted lines. Consequently, if

$\lim_{x \to \underline{x}^+} \omega^z(x, \underline{x}, r) \geq 0$ and $\lim_{x \to \bar{x}^-} \omega^z(x, \bar{x}, r) \geq 0$ hold, the intersections of each compatibility curve and the sustainability curve move downward. Thus, the upper and lower bounds of the equilibrium rents decrease. However, the intersections may move to the left or to the right depending on the relative magnitudes of the shifts of these curves. Proposition 3 nevertheless demonstrates that the sustainability curve shifts relatively more compared to the compatibility curves, and as a result, the intersections move to the right. Therefore, the upper and lower bounds of the equilibrium sizes of dwellings increase.

4 Equilibrium for specified preferences

In this subsection, we explore the equilibria assuming a special functional form for the utility function as well as the amenity function. The purpose is not only to illustrate the above findings in a familiar setting but also to derive further implications. We consider a utility function $u(x, z, a) = x^\theta z^{1-\theta} a^\nu$ for $\theta \in (0, 1)$ and $\nu \in (0, 1)$, and a minimum-value amenity function $a(x_1, ..., x_n) = \kappa\left(\min[x_1, ..., x_n]\right)^\eta$ for $\kappa > 0$ and $\eta > 0$. The following function summarizes necessary information:

$$v(x, x^0, r) = \kappa^\nu x^\theta z^{1-\theta} \left(\min[x, x^0]\right)^\phi,$$

where $\phi \equiv \eta\nu > 0$.

The associated second-order derivative is given by

$$v_{11}(x, x^0, r) = \begin{cases} -\dfrac{v(x, x^0, r)}{x^2 z^2} q(x) & \text{if } x < x^0 \\ -\dfrac{v(x, x^0, r)\theta(1-\theta)y^2}{x^2 z^2} < 0 & \text{if } x > x^0, \end{cases}$$

where $q(x) \equiv \theta(1-\theta)y^2 + 2\phi z(rx - \theta y) + \phi(1-\phi)z^2$. Here, z is evaluated at $y - rx$ and the argument of y, i.e., d, is dropped from the expression for the sake of exposition (these two qualifications are applicable throughout the rest of the paper). To judge the sign of $q(x)$, note, first, that $q(x)$ takes a minimum value in the interval $x \in [0, y/r]$ either at $x = y/r$ or at $x = 0$, because $q(x)$ is strictly concave in x. Second, $q(y/r) = \theta(1-\theta)y^2 > 0$. Third and last, $q(0) = (1-\theta-\phi)(\theta+\phi)y^2$, which is positive if

$$\theta + \phi < 1. \tag{12}$$

These observations altogether imply that $v_{11}(x, x^0, r) < 0$ for any $x < x^0$ and Assumption 1 is satisfied, if (12) holds. The qualification requires that when a consumer's dwelling is smaller than the neighbors', her marginal utility of housing consumption inclusive of the own amenity improvement effect be sufficiently small. In the rest of the paper, we assume this.

The compatibility set is given by

$$[\underline{x}(r^*), \bar{x}(r^*)] = \left[\frac{\theta y}{r^*}, \frac{(\theta+\phi)y}{(1+\phi)r^*}\right].$$

Three observations follow. First, because $x^* \geq \underline{x}(r^*)$, the share of spending on housing is no smaller than θ. In the standard model of a consumer behavior without amenity goods, the share would become equal to θ. In our model, however, the share can exceed it because the rise in the share enhances the utility through the amenity improvement. Second, since $\phi > 0$ and $\theta \in (0,1)$, $\theta < (\theta+\phi)/(1+\phi)$; and therefore, the interval has a positive measure. Third and finally, both $\underline{x}(\cdot)$ and $\bar{x}(\cdot)$ are decreasing functions.

Moreover, the sustainability curve is derived by solving the following equation:

$$(x^*)^{\theta+\phi} (y - r^* x^*)^{1-\theta} = \bar{u}', \tag{13}$$

where $\bar{u}' \equiv \bar{u}/\kappa^{\nu}$. Applying (11) yields

$$s'(r^*) = \frac{(1-\theta)(x^*)^2}{(\theta+\phi)y - (1+\phi)r^* x^*}.$$

The denominator is positive if $x^* \leq \bar{x}(r^*)$. Therefore, the part of the sustainable curve in the compatibility set is indeed upward sloping.

Then, substituting $\underline{x}(r^*)$ ($\bar{x}(r^*)$, resp.) into x^* in (13) and solving the resulting equation for r^*, we can derive x^l and r^l (x^u and r^u, resp.):

$$(x^l, r^l) = \left(K_x^l y^{-\Theta\Psi},\ K_r^l y^{\Phi\Psi}\right) \quad \text{and} \quad (x^u, r^u) = \left(K_x^u y^{-\Theta\Psi},\ K_r^u y^{\Phi\Psi}\right),$$

where $\Theta \equiv 1-\theta$, $\Phi \equiv 1+\phi$, $\Psi \equiv (\theta+\phi)^{-1}$ and all the K's are positive constants.[7] Several observations follow. First, x^u (r^u, resp.) is proportional to x^l (r^l, resp.); we denote the proportional coefficient by $C_x \equiv x^u/x^l$ ($C_r \equiv r^u/r^l$, resp.). Second, $\mathrm{d}x^l/\mathrm{d}d > 0$, $\mathrm{d}x^u/\mathrm{d}d > 0$, $\mathrm{d}r^l/\mathrm{d}d < 0$ and $\mathrm{d}r^u/\mathrm{d}d < 0$. Third, let us take a look at an equilibrium density at each point in a city, which is an inverse of the size of a dwelling. Note that $\mathrm{d}^2(1/x^l)/\mathrm{d}d^2$ and $\mathrm{d}^2(1/x^u)/\mathrm{d}d^2$ are positive if the following two conditions are satisfied. For one thing, θ and/or ϕ is sufficiently small that $2\theta+\phi < 1$, or equivalently, $\Theta\Psi > 1$. Furthermore, $t''(d)$ is negative or sufficiently small when it is positive.[8] In other words, the lower and upper bounds of the equilibrium density become *convex* functions of the distance from a city center as long as consumers' inclination for housing and amenities is not too strong and the commuting costs do not increase with distance too rapidly. This result is consistent with the well-known empirical fact of a declining density gradient since Clark (1951).[9]

How do various factors affect the range of the multiple equilibria? To see this, we look at the changes in the ratios of upper and lower bounds, C_x and C_r.

First, let us examine a change in θ. For one thing, it is readily verified that $\mathrm{d}\ln C_x/\mathrm{d}\theta = -(1+\phi)\ln(1+\phi)/(\theta+\phi)^2 < 0$. Furthermore, note that $\mathrm{d}\ln C_r/\mathrm{d}\theta = k_1/\left[\theta(\theta+\phi)^2\right]$, where

[7] Note that $K_x^l \equiv \left[\bar{u}/\Theta^{\Theta}\right]^{\Psi}$, $K_r^l \equiv \theta\left[\Theta^{\Theta}/\bar{u}\right]^{\Psi}$, $K_x^u \equiv \Phi^{-1}\left[\bar{u}\Phi^{\Phi}/\Theta^{\Theta}\right]^{\Psi}$ and $K_r^u \equiv \Psi^{-1}\left[\Theta^{\Theta}/\bar{u}\Phi^{\Phi}\right]^{\Psi}$.

[8] We obtain

$$\frac{\mathrm{d}^2(1/x^l)}{\mathrm{d}d^2} = \frac{\Theta\Psi}{x^l y}\left[\frac{(\Theta\Psi-1)\{t'(d)\}^2}{y} - t''(d)\right] \quad \text{and} \quad \frac{\mathrm{d}^2(1/x^u)}{\mathrm{d}d^2} = C_x \frac{\mathrm{d}^2(1/x^l)}{\mathrm{d}d^2}.$$

[9] For surveys, see Mills and Tan (1980), McDonald (1989) and Anas et al. (1998) among others.

$k_1 \equiv -\phi(\theta+\phi)+\theta(1+\phi)\ln(1+\phi)$. Because $k_1 < \theta(1+\phi)\big[\ln(1+\phi)-\phi\big] < 0$, we conclude that $\mathrm{d}\ln C_r/\mathrm{d}\theta < 0$.[10] Therefore, when consumers put a relatively higher value on housing compared to the composite good, the range of multiple equilibria, measured by either the price or the amount of housing, is narrower.

Second, we examine a change in ϕ. Note that $\mathrm{d}\ln C_x/\mathrm{d}\phi = k_2(1-\theta)/\big[(1+\phi)(\theta+\phi)^2\big]$, where $k_2 \equiv \theta+\phi-(1+\phi)\ln(1+\phi)$. The sign of k_2 is ambiguous. It is positive when θ is sufficiently large and ϕ is sufficiently small (it increases with θ and decreases with ϕ). Indeed, we can show that it is unambiguously positive if $\theta > \phi$, or, in other words, if amenities are not too important for consumers compared to housing.[11] In such a case, $\mathrm{d}\ln C_x/\mathrm{d}\phi > 0$: as amenities become relatively more important, the range of the amounts of housing consumption associated with multiple equilibria expands. On the other hand, the effect upon C_r is unambiguously positive: $\mathrm{d}\ln C_r/\mathrm{d}\phi = (1-\theta)\ln(1+\phi)/(1+\phi) > 0$. Thus, the rise in the relative importance of amenities results in a wider range of the housing prices associated with multiple equilibria.

One of the other variables worth exploring is the share of the expenditure on housing, $h \equiv rx/y$. The lower and upper bounds of the share become equal to $h^l \equiv r^l x^l/y = \theta$ and $h^u \equiv r^u x^u/y = (\theta+\phi)/(1+\phi)$, respectively. Both bounds increase with θ and the upper bound increases with ϕ. The ratio of the two bounds of the share, $C_h \equiv h^u/h^l = (\theta+\phi)/\big[\theta(1+\phi)\big]$, decreases with θ and increases with ϕ.

5 Concluding remarks

This paper studies the sizes and rents of dwellings determined as a result of the choices of consumers whose utility depends on the level of landscape amenities in their neighborhood. The level is, in turn, determined by the sizes of dwellings there. It is shown that there emerges a possibility of multiple equilibria if the effect of each consumer's choice upon landscape amenities is asymmetric and the negative effect of consuming a slightly smaller dwelling dominates the positive effect of consuming a slightly larger dwelling. If the landscape amenities are symmetric, in contrast, there is no multiple equilibria, provided that some regularity conditions are satisfied. One implication of the multiple equilibria is that the land rent gradient may become positive.

[10] One can prove that $\beta(\phi) \equiv \ln(1+\phi)-\phi < 0$ for any $\phi > 0$ from two observations: first, $\beta(0) = 0$ and second, $\beta'(\phi) = -\phi/(1+\phi) < 0$ for any $\phi > 0$.

[11] This is explained as follows. Note that $k_2 = \theta - \phi + \gamma(\phi)$ where $\gamma(\phi) \equiv 2\phi - (1+\phi)\ln(1+\phi)$. It is readily verified that $\gamma(\phi) > 0$ for any $\phi < 1$ since $\gamma(0) = 0$ and $\gamma'(\phi) = 1 - \ln(1+\phi) > 0$ for $\phi < 1$. Therefore, $\theta > \phi$ implies that $k_2 > 0$.

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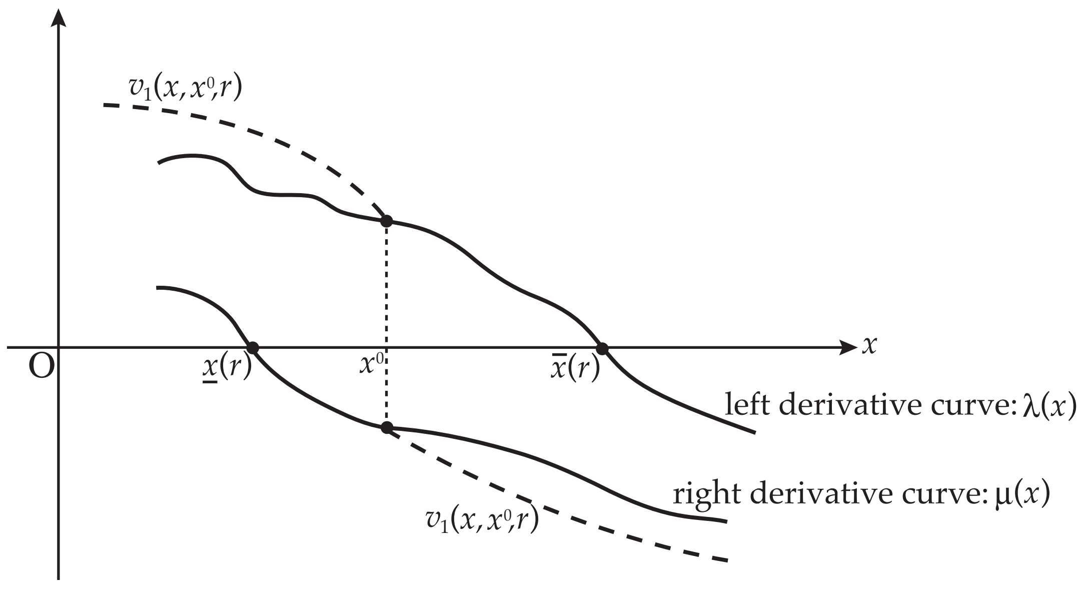

Figure 1: Left and right derivative curves

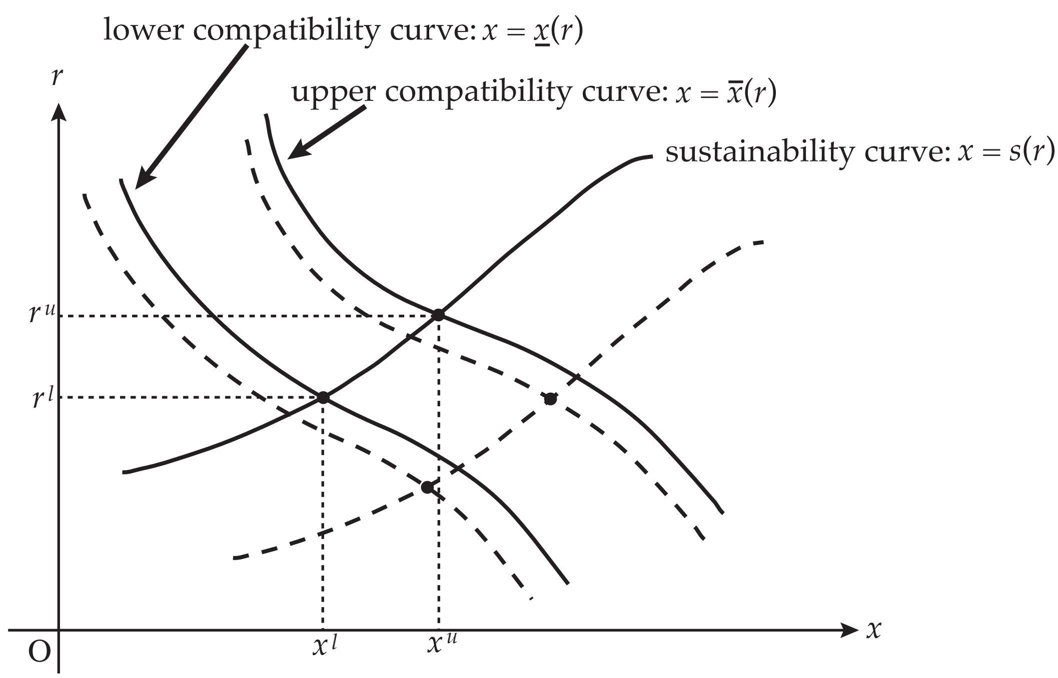

Figure 2: Compatibility curves and sustainability curve